Harmonic distances, centralities and systemic stability in heterogeneous interbank networks

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The views expressed in this presentation are those of the author and do not necessarily represent official positions of the MNB.

Outline

Research question: how do systemic importance measures perform in interbank networks in terms of direct contagion?

Fundamental papers in the topic:

- Eisenberg and Noe (2001): solution of the payment equilibrium problem, fictitious default algorithm.
- Allen and Gale (2000), Freixas et al. (2000): more diversified interbank liabilities lead to a more resilient system to the default of any bank.

Most related works:

- Acemoglu et al. (2015a): vulnerability of financial networks and a new special metric of systemic importance
- Acemoglu et al. (2015b): general approach to economic networks, derivation of systemic importance measures
- Alter et al. (2015), Fink et al. (2015): empirical evidence on the usefulness of centrality measures

This work:

- presents numerical performance results for common and recently proposed systemic importance measures
- proposes a slight improvement for the harmonic distance of Acemoglu et al. (2015b), provides explicit analytical solution
- underpins that the usage of centrality measures is completely misleading in some situations: the structure of the network drives the performance of measures
- presents network measures as systemic stress indicators in a factor model approach, explained variance increases

Basic 'off-the-shelf' centrality measures

adjacency matrix: $\mathbf{A}$, undirected, weighted matrix: $\mathbf{W}$.

- degree: $d_i = \sum_{j=1}^{n} a_{i,j} = (\mathbf{A} \cdot \mathbf{1})_i$
- weighted degree: $w_i = \sum_{j=1}^{n} w_{i,j} = (\mathbf{W} \cdot \mathbf{1})_i$
- closeness: $c_i = \frac{1}{\max_j d(i,j)}$, where $d(i,j)$ denotes distance between node i and j, i.e. the minimum length of paths between them.
- betweenness: the number of shortest paths that contain a given node i. Paths of length 1 are excluded.
- eigenvector centrality: $\mathbf{A} \cdot \mathbf{v} = \lambda \cdot \mathbf{v}$
- Bonacich centrality: $b_i(\alpha, \beta) = \sum_j \alpha + \beta \cdot a_{i,j} \cdot b_j(\alpha, \beta)$, leading to $\mathbf{b}(\alpha, \beta) = \alpha \cdot (\mathbf{I} - \beta \mathbf{A})^{-1} \cdot \mathbf{1} = \alpha \cdot \mathbf{B} \cdot \mathbf{1}$.

Theoretical connections between these measures

- degree: d_i is the number of steps (paths of length 1) from node i
- weighted degree: w_i is the number of weighted steps from node i
- betweenness: the number of shortest paths that contain a given node i. Paths of length 1 are excluded.
- Bonacich centrality: $\mathbf{b}(\alpha, \beta) = \alpha \cdot \sum_{k=0}^{\infty} \beta^k \mathbf{A}^k \cdot \mathbf{1}$: the expected number of paths from node i, probability of a step is β
- eigenvector centrality is a limit of Bonacich: $\lim_{\beta \to \left(\frac{1}{\lambda_1}\right)_-} (1 - \beta\lambda_1) \cdot \mathbf{b}(1, \beta) \propto \mathbf{v}$.

Adjacency eigenvector and Bonacich centrality seemed to be the best performing measures in empirical papers of Alter et al. (2015) and Fink et al. (2015).

Economic networks and contagion

Following Acemoglu et al. (2015a); Acemoglu et al. (2015b).

Generalized economic networks: An economy of n agents $\{1, 2, \ldots, n\}$. An agent i has a state x_i ($x_i \in \mathbb{R}, i \in N$) which can be output, investment or liabilities. For an f continuous and increasing function (interaction function) let $x_i = f\left(\sum_{j=1}^{n} w_{i,j} \cdot x_j + \varepsilon_i\right)$.
Equilibrium exists and is unique. The *macro state* of the economy is $y = g\left(h(x_1) + h(x_2) + \ldots + h(x_n)\right)$.

Financial contagion: $\mathbf{x}^* = [\min\{\mathbf{Q}\mathbf{x}^* + \mathbf{e}, \mathbf{y}\}]^+$, $f(x) = [\min\{x + e, y\}]^+$.
$\mathbf{Q} = \{q_{i,k}\}_{i,k=1}^{n} = \left\{\frac{y_{i,k}}{y_k}\right\}_{i,k=1}^{n}$. $\mathbf{y}$: vector of total liabilities, $q_{i,k} \cdot y_k = y_{i,k}$, $\mathbf{x}^*$: outgoing payments in equilibrium.

Taylor series expansion

- **Bonacich centrality:** How a shock to agent p affects the state of agent i: $\frac{\partial x_i}{\partial \varepsilon_p}\Big|_{\boldsymbol{\varepsilon}=\mathbf{0}} = \alpha \cdot b_{i,p} \Rightarrow \mathbf{x} = \alpha \cdot \mathbf{B} \cdot \boldsymbol{\varepsilon}$.
or the macro state: $\frac{\partial y}{\partial \varepsilon_p} = \sum_{i=1}^{n} b_{i,p}$.

- **concentration centrality:** If one takes the second order approximation: $con_p = \mathrm{stdev}(b_{1,p}, \ldots, b_{n,p})$

  Node i is said to be systemically more important than j if $y_{(i)} > y_{(j)}$, where $y_{(i)}$ denotes the macro state when i is hit with a negative shock. Precisely, in case of concave interaction function, institution i is systemically more important than j if $con_i > con_j$.

Harmonic distance

Assume that bank j defaults in the network.
In homogeneous networks (identical liabilities): Mean hitting time of a random walk on a graph from vertex i to j is the expected number of steps of a random walk from i until it reaches j: $h_{i,j} = \mathbf{E}_i(\tau_j) = 1 + \sum_{k \neq j} \left(\frac{y_{i,k}}{y} \right) \cdot h_{k,j}, h_{i,i} = 0.$

In heterogeneous networks: Scaling of banks: $\theta_i \cdot y = y_i$. The harmonic distance of bank j to bank i is given by
$h_{i,j} = \theta_i + \sum_{k \neq j} \left(\frac{y_{i,k}}{y_k} \right) \cdot h_{k,j}, h_{i,i} = 0.$

Banks that are closer in harmonic distance to the defaulted bank are more vulnerable to distress.

Extended harmonic distance

A new extension of harmonic distance: different scaling for liquid assets and total liabilities.
$h_{i,j} = e_i + \sum_{k \neq j} \left(\frac{y_{i,k}}{y_k} \right) \cdot h_{k,j}$, where e_i is the liquid assets of bank i.

Proposition

Suppose that bank j is hit with a negative shock $\varepsilon > \sum_{i=1}^{n} e_i$. Then

1. *bank j defaults*
2. *all other banks also default if and only if $h_{i,j} < y_i$ for all i.*

Definitions are recursive, how to calculate pairwise harmonic distances? In matrix form:

$(\mathbf{I} - \mathbf{Q}) \cdot \mathbf{H} = \mathbf{E} - (\sum_{i=1}^{n} y_i) \cdot \mathbf{I}$, but $(\mathbf{I} - \mathbf{Q})$ is not invertible!

Extended harmonic distance

Proposition

The matrix $\mathbf{H} = \{h_{i,j}\}_{i,j=1}^{n}$ *of pairwise size-adjusted harmonic distances is explicitly given by*
$\mathbf{H} = -\left(\sum_{i=1}^{n} e_i\right) \cdot \left(\mathbf{I} - \mathbf{Q} + \frac{1}{\sum_{i=1}^{n} e_i} \cdot \mathbf{E}\right)^{-1} + \mathbf{D}$, *if and only if there is no non-borrowing node in the directed network.*

($\mathbf{d}_i = \left[-\mathbf{v}_0 \cdot \frac{m_{i,i}}{v_{0,i}}\right]$ is the ith column of $\mathbf{D}$ and $\mathbf{v}_0$ is the eigenvector of $(\mathbf{I} - \mathbf{Q})$ corresponding to 0 eigenvalue.)

Straightforward systemic importance measure of j is then defined by $\sum_{i=1}^{n} h_{i,j}$.

How to compare all these measures?

Idea: the systemic importance of a bank in terms of direct contagion is the aggregate loss induced to the system by its default.

1. generate a huge number of networks with similar structure (following Soramäki and Cook (2013))
2. induce initial defaults one-by-one and compute the payment equilibrium $\mathbf{x}^*$ (following Eisenberg and Noe (2001)) and aggregate losses $= \sum_{i=1}^{n}(y_i - x_i^*)$
3. compute correlations of losses to centralities for all networks in a fixed parameter set

Artificial interbank networks

FOR $i = 1..n_0$ (add initial banks/nodes)
    SET $h_i = 1$
END FOR
    SET $active = 0$ (initial number of active banks in the network)
    SET $k = n_0 + 1$ (first new bank)
WHILE $active < n$
    FOR $l = 1..m$ (average number of payments per bank)
        SELECT random sender $i \in \{1, \ldots, k\}$ such that bank i has the probability $\frac{h_i}{\sum_l h_l}$ of
        SET $h_i = h_i + \alpha$ (update preferential attachment strength)
        SELECT random receiver $j \in \{1, \ldots, k\}$ such that bank j has the probability $\frac{h_j}{\sum_l h_l}$ of being selected as recipient of the payment
        SET $h_j = h_j + \alpha$ (update preferential attachment strength)
        SET $y_{j,i} = y_{j,i} + 1$ (create payment/link)
    END FOR
    IF $k \leq n$ SET $h_k = 1$ AND SET $k = k + 1$ (create new bank/node)
    SET *active* as the number of nodes sending or receiving any payments
END WHILE

- n: desired number of banks
- n_0: initial number of banks
- α: preferential attachment parameter
- m: number of edges attached at an iteration step
- h_i: 'strength' of node i
- edge weights are log-normally distributed proportionally to the minimum of in-degree and out-degree
- liquid assets are determined to have no contagion without the default of any banks: scaling with c

Transformation of harmonic distances

Instead of $\sum_{i=1}^{n} h_{i,j}$, use $\frac{1}{\sum_{i=1}^{n} h_{i,j}}$.

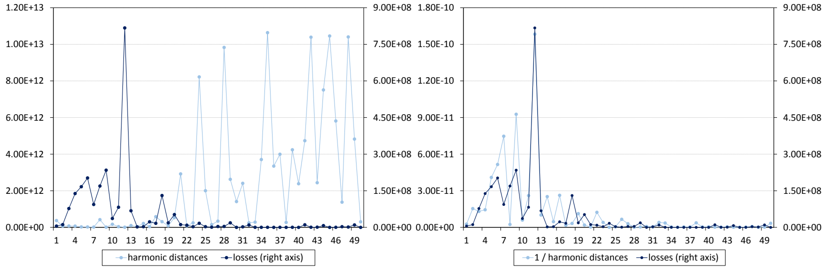

Table: $n = 50,\ n_0 = 5,\ m = 4,\ \alpha = 0.1$

Correlations on 1000 networks for fixed parameters

harmonic distances:

c	1		2		3	
α	avg.corr.	std.dev.	avg.corr.	std.dev.	avg.corr.	std.dev.
0.1	0.406	0.215	0.423	0.219	0.410	0.215
0.2	0.524	0.229	0.529	0.226	0.531	0.236
0.4	0.665	0.223	0.675	0.223	0.669	0.236
0.6	0.703	0.215	0.721	0.220	0.733	0.227

extended harmonic distances:

c	1		2		3	
α	avg.corr.	std.dev.	avg.corr.	std.dev.	avg.corr.	std.dev.
0.1	0.443	0.229	0.460	0.236	0.443	0.227
0.2	0.579	0.247	0.587	0.242	0.587	0.252
0.4	0.733	0.224	0.738	0.226	0.732	0.239
0.6	0.773	0.207	0.778	0.215	0.794	0.217

weighted degree:

c	1		2		3	
α	avg.corr.	std.dev.	avg.corr.	std.dev.	avg.corr.	std.dev.
0.1	0.783	0.063	0.793	0.070	0.796	0.067
0.2	0.805	0.071	0.816	0.073	0.817	0.072
0.4	0.839	0.076	0.848	0.077	0.846	0.082
0.6	0.849	0.079	0.861	0.080	0.866	0.080

eigenvector:

c	1		2		3	
α	avg.corr.	std.dev.	avg.corr.	std.dev.	avg.corr.	std.dev.
0.1	0.746	0.078	0.757	0.078	0.757	0.078
0.2	0.782	0.076	0.792	0.079	0.792	0.079
0.4	0.821	0.080	0.830	0.079	0.831	0.084
0.6	0.836	0.084	0.843	0.086	0.849	0.083

Table: Average correlation of centrality measures and losses generated by the failure of single nodes and standard deviation of correlations.

Correlations on 1000 networks for fixed parameters

Bonacich:

c	1		2		3	
α	avg.corr.	std.dev.	avg.corr.	std.dev.	avg.corr.	std.dev.
0.1	0.397	0.188	0.380	0.187	0.375	0.187
0.2	0.466	0.212	0.450	0.192	0.451	0.198
0.4	0.538	0.219	0.511	0.208	0.505	0.214
0.6	0.505	0.214	0.509	0.225	0.503	0.221

concentration:

c	1		2		3	
α	avg.corr.	std.dev.	avg.corr.	std.dev.	avg.corr.	std.dev.
0.1	0.386	0.193	0.369	0.193	0.364	0.192
0.2	0.456	0.216	0.441	0.198	0.441	0.201
0.4	0.527	0.222	0.502	0.211	0.490	0.219
0.6	0.490	0.219	0.493	0.225	0.489	0.220

closeness:

c	1		2		3	
α	avg.corr.	std.dev.	avg.corr.	std.dev.	avg.corr.	std.dev.
0.1	0.431	0.095	0.422	0.096	0.425	0.092
0.2	0.403	0.100	0.402	0.101	0.401	0.102
0.4	0.364	0.117	0.355	0.119	0.355	0.120
0.6	0.316	0.128	0.314	0.130	0.315	0.128

betweenness:

c	1		2		3	
α	avg.corr.	std.dev.	avg.corr.	std.dev.	avg.corr.	std.dev.
0.1	0.373	0.168	0.369	0.162	0.367	0.162
0.2	0.345	0.168	0.357	0.173	0.356	0.177
0.4	0.300	0.182	0.289	0.183	0.292	0.189
0.6	0.230	0.188	0.232	0.191	0.234	0.188

Table: Average correlation of centrality measures and losses generated by the failure of single nodes and standard deviation of correlations.

Correlations on 10000 networks when liquid assets are uniformly random, $c \in (1, 3)$

harmonic distances

α	avg.corr.	std.dev.
0.1	0.416	0.217
0.2	0.540	0.234
0.4	0.666	0.232
0.6	0.722	0.223

extended harmonic distances

α	avg.corr.	std.dev.
0.1	0.450	0.231
0.2	0.591	0.248
0.4	0.724	0.235
0.6	0.778	0.219

weighted degree

α	avg.corr.	std.dev.
0.1	0.794	0.071
0.2	0.819	0.075
0.4	0.847	0.079
0.6	0.864	0.080

eigenvector

α	avg.corr.	std.dev.
0.1	0.759	0.078
0.2	0.796	0.078
0.4	0.828	0.082
0.6	0.846	0.083

Bonacich

α	avg.corr.	std.dev.
0.1	0.377	0.185
0.2	0.455	0.197
0.4	0.502	0.214
0.6	0.508	0.229

concentration

α	avg.corr.	std.dev.
0.1	0.365	0.190
0.2	0.444	0.202
0.4	0.491	0.217
0.6	0.494	0.229

closeness

α	avg.corr.	std.dev.
0.1	0.427	0.094
0.2	0.402	0.100
0.4	0.356	0.117
0.6	0.313	0.127

betweenness

α	avg.corr.	std.dev.
0.1	0.433	0.164
0.2	0.428	0.181
0.4	0.293	0.185
0.6	0.303	0.195

Table: Average correlation of centrality measures compared to losses generated by the failure of single nodes, randomized liquid assets.

'Mean' behaviour

Centralities and losses are averaged for a given parameter set. If $c_{i,t}$ is a centrality of bank i in network t, then the average centrality of bank i will be $\sum_{t=1}^{T} \frac{c_{i,t}}{T}$.

harmonic distances

c	1	2	3
α	corr.	corr.	corr.
0.1	0.993	0.994	0.991
0.2	0.985	0.979	0.987
0.4	0.947	0.915	0.947
0.6	0.880	0.839	0.900

extended harmonic distances

c	1	2	3
α	corr.	corr.	corr.
0.1	0.993	0.995	0.991
0.2	0.986	0.980	0.988
0.4	0.950	0.916	0.949
0.6	0.877	0.811	0.881

weighted degree

c	1	2	3
α	corr.	corr.	corr.
0.1	0.998	0.999	0.998
0.2	0.999	0.999	1.000
0.4	0.999	0.999	0.998
0.6	0.996	0.997	0.995

eigenvector

c	1	2	3
α	corr.	corr.	corr.
0.1	0.995	0.996	0.996
0.2	0.999	0.998	0.999
0.4	0.999	0.998	0.999
0.6	0.997	0.997	0.997

Bonacich

c	1	2	3
α	corr.	corr.	corr.
0.1	0.993	0.993	0.992
0.2	0.996	0.993	0.995
0.4	0.997	0.996	0.995
0.6	0.992	0.995	0.994

concentration

c	1	2	3
α	corr.	corr.	corr.
0.1	0.993	0.993	0.992
0.2	0.996	0.993	0.995
0.4	0.997	0.996	0.995
0.6	0.992	0.995	0.994

closeness

c	1	2	3
α	corr.	corr.	corr.
0.1	0.864	0.847	0.853
0.2	0.800	0.798	0.793
0.4	0.719	0.710	0.714
0.6	0.651	0.635	0.662

betweenness

c	1	2	3
α	corr.	corr.	corr.
0.1	0.841	0.819	0.817
0.2	0.726	0.726	0.724
0.4	0.603	0.586	0.600
0.6	0.511	0.477	0.512

Table: Correlation of averaged network measures and average induced losses.

Results on complete networks

1000 networks, edge weights are log-normally distributed as before. Betweeness is 0, closeness is constant by definition.

	c 1 avg.corr.	std.dev.	2 avg.corr.	std.dev.	3 avg.corr.	std.dev.
harmonic distances	0.067	0.151	0.024	0.148	0.020	0.141
extended harmonic distances	0.168	0.154	0.136	0.153	0.129	0.146
weighted degree	0.654	0.087	0.680	0.076	0.677	0.077
eigenvector	0.642	0.095	0.672	0.076	0.669	0.075
Bonacich	0.040	0.151	0.002	0.147	-0.001	0.142
concentration	-0.053	0.143	0.029	0.143	-0.029	0.150
closeness	0	0	0	0	0	0
betweenness	N/A	N/A	N/A	N/A	N/A	N/A

Table: Correlations for complete networks.

Even eigenvector and weighted degrees are poor.

Behaviour in a real financial network

Hungarian unsecured interbank lending network, weekly aggregations to obtain connected components.

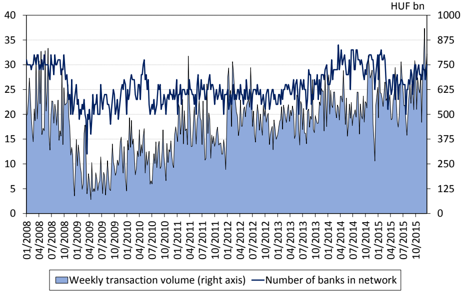

Figure: Number of banks and transaction volume in weekly networks

Mostly scale-free (Clauset et al. (2009)):

	no. of networks	p-value			γ		
		min	max	avg	min	max	avg
scale-free	327	0.100	0.993	0.477	1.886	3.344	2.906
non-scale-free	78	0.000	0.098	0.038	1.499	3.321	2.117

Table: Accepted (scale-free) and rejected (non-scale-free) networks' p-values and γ parameters of degree distributions

Behaviour in a real financial network

	avg.	std.dev.
Corr (WD, Eig)	0.704	0.076
Corr $(1/HD, WD)$	0.079	0.187
Corr $(1/HD, Eig)$	0.488	0.197
Corr (B, WD)	-0.041	0.142
Corr (B, Eig)	0.361	0.196
Corr (C, WD)	0.019	0.208
Corr (C, Eig)	0.189	0.217

Table: Average correlations and standard deviations across institutions.

Corr (WD, Eig)	0.881
Corr $(1/HD, WD)$	0.847
Corr $(1/HD, Eig)$	0.937
Corr (B, WD)	0.790
Corr (B, Eig)	0.922
Corr (C, WD)	0.259
Corr (C, Eig)	0.177

Table: Correlations on the averaged network, across institutions.

WD: weighted degrees, Eig: eigenvector centralities, 1/HD: the reciprocal of harmonic distances, B: Bonacich centralities, C: concentration centralities

Behaviour in a real financial network

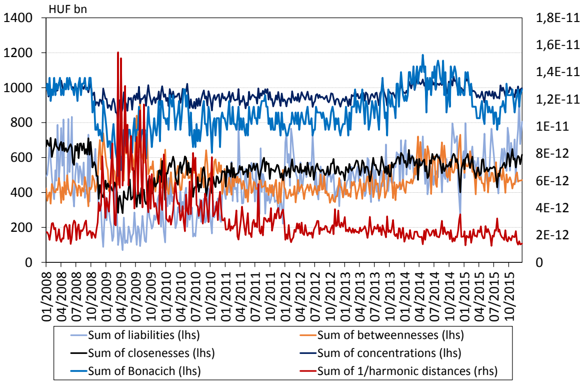

Figure: Centrality measures and the sum of all harmonic distances in the network

Which one is better for systemic stress indication? $\Rightarrow$ static factor model of financial variables

Performances in a factor model

Hałaj and Kok (2013) suggested to include network measures in a systemic stress index like CISS (Holló et al. (2012)). I use the baseline model of Szendrei and Varga (2017).

government bond market	bond yields (3-month and 10-year) CDS (5-year bond)
interbank market	BUBOR (3-month) HUFONIA overnight rate HUFONIA trading volume
banking sector	bank PDs: from market price (Merton model) *network measure*
FX market	bid-ask spreads: HUF/EUR + HUF/USD volatilities: HUF/EUR, HUF/USD, HUF/GBP, HUF/CHF
capital market	CMAX: BUX, BUMIX, CETOP20, DAX implied volatility: VDAX

Table: Variables in the factor model (Szendrei and Varga, 2017).

$\mathbf{y}_t = \lambda \cdot \mathbf{f}_t + \epsilon_t, \mathbf{f}_t \sim N(\mathbf{0}, \mathbf{I}_q)$, $\epsilon_t \sim N(\mathbf{0}, \mathbf{\Sigma})$ are iid, λ is a $n \times q$ matrix of factor loadings. The number of variables is $n = 19$ and the number of factors is $q = 4$. **Explained variance increases by approximately 2.7%.**

Explanation of results

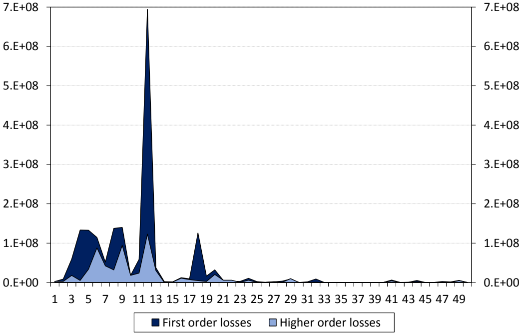

Figure: First order and higher order losses induced by the initial default of banks

- in a typical scale-free network, first order losses dominate higher order losses
- in complete networks, higher order losses are larger
- linearizing the payment equilibrium is inappropriate
- the default of all banks is not likely

Conclusions

- application of well-known centrality measures in the literature might be misleading: the structure of the network is important
- even recently proposed measures like harmonic distances and concentration centrality couldn't outperform the above: different linearizations of the payment equilibrium equations are not useful
- extended harmonic distance performed slightly better than harmonic distance
- performances are very good on averaged networks: variance disappears
- network measures are useful in systemic stress indication according to a factor model

Thank you for your attention!

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