On the Fit of New-Keynesian Models

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Abstract

The paper provides new tools for the evaluation of DSGE models, and applies them to a large-scale New Keynesian dynamic stochastic general equilibrium (DSGE) model with price and wage stickiness and capital accumulation. Specifically, we approximate the DSGE model by a vector autoregression, and then systematically relax the implied cross-equation restrictions and document how the model fit changes. Furthermore, we study the nature of the misspecification by comparing the DSGE model's impulse responses to structural shocks with those obtained after relaxing the model restrictions. We find that the degree of misspecification in large-scale DSGE models is no longer so large to prevent their use in day-to-day policy analysis, yet it is not small enough to be ignored. (JEL C11, C32, C53)

KEY WORDS: Bayesian Analysis, DSGE Models, Model Evaluation, Vector Autoregressions

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1 Introduction

Dynamic stochastic general equilibrium (DSGE) models are not just attractive from a theoretical perspective, but they are also emerging as useful tools for forecasting and quantitative policy analysis in macroeconomics. Due to improved time series fit these models are gaining credibility in policy making institutions such as central banks. Up until recently DSGE models had the reputation of being unable to track macroeconomic time series. In fact, an assessment of their forecasting performance was typically considered futile, an exception being, DeJong, Ingram, and Whiteman (2000). Apparent model misspecifications were used as an argument in favor of informal calibration approaches to the evaluation of DSGE models along the lines of Kydland and Prescott (1982). Subsequently, researchers have developed econometric frameworks that formalize aspects of the calibration approach.[1] A common feature of many evaluation procedures is that DSGE model predictions are either implicitly or explicitly compared to those from a reference model. Much of the applied work related to monetary models has, for instance, proceeded by assessing DSGE models based on discrepancies between impulse response functions obtained from the DSGE model and those obtained from the estimation of identified vector autoregressions (VARs). However, such an evaluation is only sensible if the VAR indeed dominates the DSGE model in terms of time series fit as pointed out in Schorfheide (2000).

Smets and Wouters (2003) develop a large-scale monetary DSGE model in the New Keynesian tradition based on work by Christiano, Eichenbaum, and Evans (2005) and estimate it on Euro-area data. One of the remarkable empirical results is that posterior odds favor their DSGE model relative to VARs estimated with a fairly diffuse training sample prior. Previous studies using more stylized DSGE models always found that even simple VARs dominate DSGE models. On the methodological side, Smets and Wouters' finding challenges the practice of assessing DSGE models on their ability to reproduce VAR impulse response functions without carefully documenting that the VAR indeed fits better than the DSGE model. On the substantive side, it poses the question whether researchers from now on have to be less concerned about misspecification of DSGE models.

The contributions of our paper are twofold, one methodological and the other substantive. First, we develop a set of tools that is useful to assess the time series fit of a DSGE model. In particular, we systematically relax the implied cross-coefficient restrictions of the

[1] Examples are Canova (1994), DeJong, Ingram, and Whiteman (1996), Diebold, Ohanian, and Berkowitz (1998), Geweke (1999b), Schorfheide (2000), and Dridi, Guay, and Renault (2006).

DSGE model to obtain a vector autoregressive specification that is guaranteed to fit better than the DSGE model yet simultaneously stays as close as possible to the DSGE restrictions. We use this specification as a benchmark to characterize and understand the degree of misspecification of the DSGE model. Second, we apply these tools to a variant of Smets and Wouters' model and document its fit and forecasting performance based on post-war U.S. data. We find that model misspecification is still a concern.

Our model evaluation approach is related to work on DSGE model priors for VARs by Ingram and Whiteman (1994) and Del Negro and Schorfheide (2004), as well as the idea of indirect inference developed by Gourieroux, Monfort, and Renault (1993), Smith (1993), and recently applied in a Bayesian setting by Gallant and McCulloch (2004). We use the VAR as an approximating model for the DSGE model and construct a mapping from the DSGE model to the VAR parameters. This mapping leads to a set of cross-coefficient restrictions for the VAR. Deviations from these restrictions are interpreted as evidence for DSGE model misspecification. In particular, we specify a prior distribution for deviations from the DSGE model restrictions. The prior tightness is scaled by a hyperparameter λ. The values $\lambda = \infty$ and $\lambda = 0$ correspond to the two polar cases where the cross-coefficient restrictions are strictly enforced and completely ignored (unrestricted VAR), respectively. The marginal likelihood function of $\lambda \in (0, \infty]$ provides an overall assessment of the DSGE model restrictions that is more robust and informative than a comparison of the two polar cases, which is widespread practice in literature.

We have evidence of misspecification whenever the peak $\hat{\lambda}$ of the marginal likelihood function is attained at a finite value. In this case the data suggest that fit improves if the DSGE model restrictions are relaxed. The resulting vector autoregressive specification, which we label DSGE-VAR($\hat{\lambda}$), can be used as a benchmark for evaluating the dynamics of the DSGE model. We ask the question: In which dimension do the impulse response functions change as we relax the cross-coefficient restriction? To facilitate impulse response function comparisons, we provide a coherent identification scheme for the DSGE-VAR. By coherent we mean that in the absence of DSGE model misspecification and VAR approximation error the impulse responses of DSGE model and DSGE-VAR to all structural shocks would coincide. Hence, in constructing a benchmark for the evaluation of the DSGE model we are trying to stay as close to the original specification as possible.

The empirical findings are as follows. The marginal likelihood function of the hyperparameter λ has an inverse U-shape indicating that the fit of the autoregressive system can be improved by relaxing the DSGE model restrictions. The shape of the posterior also implies that the restrictions should not be completely ignored when constructing a benchmark for the model evaluation as VARs with very diffuse priors are clearly dominated by the DSGE-VAR($\hat{\lambda}$). This finding is confirmed in the pseudo-out-of-sample forecasting experiment. According to a widely-used multivariate forecast error statistic the DSGE model and the VAR with diffuse prior perform about equally well in terms of one-step ahead forecasts, but are clearly worse than the DSGE-VAR($\hat{\lambda}$).

When comparing impulse responses between the DSGE model and the DSGE-VAR($\hat{\lambda}$) we find that the DSGE model misspecification does not translate into differences among impulse response functions to technology or monetary policy shocks. The latter result is important from a policy perspective, as it confirms that, in spite of its deficiencies, the New Keynesian DSGE model can indeed generate realistic predictions of the effects of unanticipated changes in monetary policy. However, responses to some of the other shocks differ across DSGE model and DSGE-VAR($\hat{\lambda}$), in particular in the long-run, suggesting that some low-frequency implications of the model are at odds with the data. We also use the DSGE-VAR framework to make comparisons across DSGE model specifications. In particular, we consider a version of the model without habit formation and another version without price and wage indexation. We find that the evidence from the DSGE-VAR analysis against the no-indexation specification is not nearly as strong as the evidence against the model without habit formation.

The paper is organized as follows. The DSGE model is presented in Section 2. Section 3 discusses the DSGE model evaluation framework. Section 4 describes the data. Empirical results are presented in Section 5 and Section 6 concludes.

2 The DSGE Model

This section describes the DSGE model, which is a slightly modified version of the DSGE model developed and estimated for the Euro area in Smets and Wouters (2003). In particular, we introduce stochastic trends into the model, so that it can be estimated with unfiltered time series observations. The DSGE model is based on work of Christiano, Eichenbaum, and Evans (2005) and contains a large number of nominal and real frictions. To make this paper self-contained we subsequently describe the structure of the model economy and the decision problems of the agents in the economy.

2.1 Final Goods Producers

The final good Y_t is a composite made of a continuum of intermediate goods $Y_t(i)$, indexed by $i \in [0,1]$:

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_{f,t}}} di \right]^{1+\lambda_{f,t}} \tag{1}$$

where $\lambda_{f,t} \in (0,\infty)$ follows the exogenous process:

$$\ln \lambda_{f,t} = (1-\rho_{\lambda_f}) \ln \lambda_f + \rho_{\lambda_f} \ln \lambda_{f,t-1} + \sigma_{\lambda,f} \epsilon_{\lambda,t}, \tag{2}$$

where $\epsilon_{\lambda,t}$ is an exogenous shock with unit variance that in equilibrium affects the mark-up over marginal costs. The final goods producers are perfectly competitive firms that buy intermediate goods, combine them to the final product Y_t, and resell the final good to consumers. The firms maximize profits

$$P_t Y_t - \int P_t(i) Y_t(i) di$$

subject to (1). Here P_t denotes the price of the final good and $P_t(i)$ is the price of intermediate good i. From their first order conditions and the zero-profit condition we obtain that:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} Y_t \quad \text{and} \quad P_t = \left[\int_0^1 P_t(i)^{-\frac{1}{\lambda_{f,t}}} di \right]^{-\lambda_{f,t}}. \tag{3}$$

2.2 Intermediate goods producers

Good i is made using the technology:

$$Y_t(i) = \max \left\{ Z_t^{1-\alpha} K_t(i)^\alpha L_t(i)^{1-\alpha} - Z_t \mathcal{F}, 0 \right\}, \tag{4}$$

where the technology shock Z_t (common across all firms) follows a unit root process, and where $\mathcal{F}$ represent fixed costs faced by the firm. Based on preliminary estimation results we decided to set $\mathcal{F} = 0$ in the empirical analysis. We define technology growth $z_t = \log(Z_t/Z_{t-1})$ and assume that z_t follows the autoregressive process:[2]

$$z_t = (1-\rho_z)\gamma + \rho_z z_{t-1} + \sigma_z \epsilon_{z,t}. \tag{5}$$

All firms face the same prices for their labor and capital inputs. Hence profit maximization implies that the capital-labor ratio is the same for all firms:

$$\frac{K_t(i)}{L_t(i)} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_t^k}, \tag{6}$$

[2] Smets and Wouters (2003) assume a stationary technology shock that follows an autoregressive process. Their estimate of the autocorrelation coefficient however are very close to the upper boundary of one. We therefore choose to assume a unit root process from the onset.

where W_t is the nominal wage and R_t^k is the rental rate of capital. Following Calvo (1983), we assume that in every period a fraction of firms ζ_p is unable to re-optimize their prices $P_t(i)$. These firms adjust their prices mechanically according to

$$P_t(i) = (\pi_{t-1})^{\iota_p}(\pi_*)^{1-\iota_p}, \tag{7}$$

where $\pi_t = P_t/P_{t-1}$, π_* is the steady state inflation rate of the final good, and $\iota \in [0,1]$. Those firms that are able to re-optimize prices choose the price level $\tilde{P}_t(i)$ that solves:

$$\begin{aligned} &\max_{\tilde{P}_t(i)} I\!E_t \left[\sum_{s=0}^{\infty} \zeta_p^s \beta^s \Xi_{t+s}^p \left(\tilde{P}_t(i) \left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right) - MC_{t+s} \right) Y_{t+s}(i) \right] \\ &\text{s.t. } Y_{t+s}(i) = \left(\frac{\tilde{P}_t(i) \left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right)}{P_{t+s}} \right)^{-\frac{1+\lambda_{f,t}}{\lambda_{f,t}}} Y_{t+s}, \; MC_{t+s} = \frac{\alpha^{-\alpha} W_{t+s}^{1-\alpha} R_{t+s}^{k\ \alpha}}{(1-\alpha)^{(1-\alpha)} Z_{t+s}^{1-\alpha}}, \end{aligned} \tag{8}$$

where $\beta^s \Xi_{t+s}^p$ is today's value of a future dollar for the consumers and MC_t reflects marginal costs. We consider only the symmetric equilibrium where all firms will choose the same $\tilde{P}_t(i)$. Hence from (3) we obtain the following law of motion for the aggregate price level:

$$P_t = \left[(1-\zeta_p)\tilde{P}_t^{-\frac{1}{\lambda_{f,t}}} + \zeta_p \left(\pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} P_{t-1} \right)^{-\frac{1}{\lambda_{f,t}}} \right]^{-\lambda_{f,t}}. \tag{9}$$

2.3 Labor Packers

There is a continuum of households, indexed by $j \in [0,1]$, each supplying a differentiated form of labor, $L(j)$. The labor packers are perfectly competitive firms that hire labor from the households and combine it into labor services L_t that are offered to the intermediate goods producers:

$$L_t = \left[\int_0^1 L_t(j)^{\frac{1}{1+\lambda_w}} di \right]^{1+\lambda_w}, \tag{10}$$

where $\lambda_w \in (0,\infty)$ is a fixed parameter.[3] From first-order and zero-profit conditions of the labor packers we obtain the labor demand function and an expression for the price of aggregated labor services L_t:

$$(a) \quad L_t(j) = \left(\frac{W_t(j)}{W_t} \right)^{-\frac{1+\lambda_w}{\lambda_w}} L_t \quad \text{and} \quad (b) \quad W_t = \left[\int_0^1 W_t(j)^{-\frac{1}{\lambda_w}} di \right]^{-\lambda_w}. \tag{11}$$

[3] Smets and Wouters (2003) assume that i.i.d. shocks to the degree of labor substitutability are another source of disturbance in the economy.

2.4 Households

The objective function for household j is given by:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s b_{t+s} \left[\log(C_{t+s}(j) - hC_{t+s-1}(j)) - \frac{\varphi_{t+s}}{1+\nu_l} L_{t+s}(j)^{1+\nu_l} + \frac{\chi}{1-\nu_m} \left(\frac{M_{t+s}(j)}{Z_{t+s} P_{t+s}} \right)^{1-\nu_m} \right] \tag{12}$$

where $C_t(j)$ is consumption, $L_t(j)$ is labor supply, and $M_t(j)$ is money holdings. Household's preferences display habit-persistence. The preference shifters φ_t, which affects the marginal utility of leisure, and b_t, which scales the overall period utility, are exogenous processes common to all households that evolve as:

$$\ln \varphi_t = (1-\rho_\varphi) \ln \varphi + \rho_\varphi \ln \varphi_{t-1} + \sigma_\varphi \epsilon_{\varphi,t}, \tag{13}$$

$$\ln b_t = \rho_b \ln b_{t-1} + \sigma_b \epsilon_{b,t}. \tag{14}$$

Real money balances enter the utility function deflated by the (stochastic) trend growth of the economy, so to make real money demand stationary.

The household's budget constraint written in nominal terms is given by:

$$\begin{aligned} &P_{t+s}C_{t+s}(j) + P_{t+s}I_{t+s}(j) + B_{t+s}(j) + M_{t+s}(j) \leq R_{t+s-1}B_{t+s-1}(j) + M_{t+s-1}(j) + A_{t+s-1}(j) \\ &+ \Pi_{t+s} + W_{t+s}(j)L_{t+s}(j) + \left(R^k_{t+s} u_{t+s}(j)\bar{K}_{t+s-1}(j) - P_{t+s} a(u_{t+s}(j))\bar{K}_{t+s-1}(j) \right), \end{aligned} \tag{15}$$

where $I_t(j)$ is investment, $B_t(j)$ is holdings of government bonds, R_t is the gross nominal interest rate paid on government bonds, $A_t(j)$ is the net cash inflow from participating in state-contingent securities, Π_t is the per-capita profit the household gets from owning firms (households pool their firm shares, and they all receive the same profit), and $W_t(j)$ is the nominal wage earned by household j. The term within parenthesis represents the return to owning $\bar{K}_t(j)$ units of capital. Households choose the utilization rate of their own capital, $u_t(j)$. Households rent to firms in period t an amount of effective capital equal to:

$$K_t(j) = u_t(j)\bar{K}_{t-1}(j), \tag{16}$$

and receive $R^k_t u_t(j)\bar{K}_{t-1}(j)$ in return. They however have to pay a cost of utilization in terms of the consumption good equal to $a(u_t(j))\bar{K}_{t-1}(j)$. Households accumulate capital according to the equation:

$$\bar{K}_t(j) = (1-\delta)\bar{K}_{t-1}(j) + \mu_t \left(1 - S\left(\frac{I_t(j)}{I_{t-1}(j)} \right) \right) I_t(j), \tag{17}$$

where δ is the rate of depreciation, and $S(\cdot)$ is the cost of adjusting investment, with $S(e^\gamma) = 0$, and $S''(\cdot) > 0$. The term μ_t is a stochastic disturbance to the price of investment relative to consumption, see Greenwood, Hercovitz, and Krusell (1998), which follows the exogenous process:[4]

$$\ln \mu_t = (1 - \rho_\mu) \ln \mu + \rho_\mu \ln \mu_{t-1} + \sigma_\mu \epsilon_{\mu,t}. \tag{18}$$

The households' wage setting is subject to nominal rigidities á la Calvo (1983). In each period a fraction ζ_w of households is unable to re-adjust wages. For these households, the wage $W_t(j)$ will increase at a geometrically weighted average of the steady state rate increase in wages (equal to steady state inflation π_* times the steady state growth rate of the economy e^γ) and of last period's inflation times last period's productivity $(\pi_{t-1}e^{z_{t-1}})$. The weights are $1 - \iota_w$ and ι_w, respectively. Those households that are able to re-optimize their wage solve the problem:

$$\begin{array}{ll} \max_{\tilde{W}_t(j)} & \mathbb{E}_t \sum_{s=0}^{\infty} \zeta_w^s \beta^s b_{t+s} \left[-\frac{\varphi_{t+s}}{1+\nu_l} L_{t+s}(j)^{1+\nu_l} \right] \\ \text{s.t.} & \text{Eq. (15) for } s = 0, \ldots, \infty, \text{ (11a), and} \\ & W_{t+s}(j) = \left(\Pi_{l=1}^s (\pi_* e^\gamma)^{1-\iota_w} (\pi_{t+l-1} e^{z_{t+l-1}})^{\iota_w} \right) \tilde{W}_t(j). \end{array} \tag{19}$$

We again consider only the symmetric equilibrium in which all agents solving (19) will choose the same $\tilde{W}_t(j)$. From (11b) it follows that:

$$W_t = [(1 - \zeta_w)\tilde{W}_t^{-\frac{1}{\lambda_w}} + \zeta_w((\pi_* e^\gamma)^{1-\iota_w}(\pi_{t-1}e^{z_{t-1}})^{\iota_w} W_{t-1})^{-\frac{1}{\lambda_w}}]^{-\lambda_w}. \tag{20}$$

Finally, we assume there is a complete set of state contingent securities in nominal terms, which implies that the Lagrange multiplier $\Xi_t^p(j)$ associated with (15) must be the same for all households in all periods and across all states of nature. This in turn implies that in equilibrium households will make the same choice of consumption, money demand, investment and capital utilization. Since the amount of leisure will differ across households due to the wage rigidity, separability between labor and consumption in the utility function is key for this result.

2.5 Government Policies

The central bank follows a nominal interest rate rule by adjusting its instrument in response to deviations of inflation and output from their respective target levels:

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{Y_t}{Y_t^*}\right)^{\psi_2} \right]^{1-\rho_R} e^{\sigma_R \epsilon_{R,t}}, \tag{21}$$

[4] We have also experimented with the introduction of a deterministic trend Υ^t in Equation (17). Since this added parameter does not change the results or improve the fit for our empirical specification, we set it equal to 1.

where $\epsilon_{R,t}$ is the monetary policy shock, R^* is the steady state nominal rate, Y_t^* is the target level of output, and the parameter ρ_R determines the degree of interest rate smoothing. This specification of the Taylor rule is more standard than the one in Smets and Wouters (2003), who introduce a time-varying inflation objective that varies stochastically according to a random walk. The random walk inflation target may help the model to fit the medium- and long-frequency fluctuations in inflation. In this paper, we are interested in assessing the model's fit of inflation without the extra help coming from the exogenous inflation target shocks. We set the target level of output Y_t^* in (21) equal to the trend level of output $Y_t^* = Z_t Y^*$, where Y^* is the steady state of the model expressed in terms of detrended variables.[5] The central bank supplies the money demanded by the household to support the desired nominal interest rate.

The government budget constraint is of the form

$$P_t G_t + R_{t-1} B_{t-1} + M_{t-1} = T_t + M_t + B_t, \tag{22}$$

where T_t are nominal lump-sum taxes (or subsidies) that also appear in household's budget constraint. Government spending is given by:

$$G_t = (1 - 1/g_t) Y_t, \tag{23}$$

where g_t follows the process:

$$\ln g_t = (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \sigma_g \epsilon_{g,t} \tag{24}$$

2.6 Resource Constraint

The aggregate resource constraint:

$$C_t + I_t + a(u_t)\bar{K}_{t-1} = \frac{1}{g_t} Y_t. \tag{25}$$

can be derived by integrating the budget constraint (15) across households, and combining it with the government budget constraint (22) and the zero profit conditions of both labor packers and final good producers.

[5] We also considered an alternative specification in which the central bank targets the level of output that would have prevailed in absence of nominal rigidities. Preliminary estimation results indicated that the flexible price target leads to deterioration of fit.

2.7 Model Solution

As in Altig, Christiano, Eichenbaum, and Lindé (2004) our model economy evolves along stochastic growth path. Output Y_t, consumption C_t, investment I_t, the real wage W_t/P_t, physical capital $\bar{K}_t$ and effective capital K_t all grow at the rate Z_t. Nominal interest rates R_t, inflation π_t, and hours worked L_t are stationary. The model can be rewritten in terms of detrended variables. We find the steady states for the detrended variables and use the method in Sims (2002) to construct a log-linear approximation of the model around the steady state. We collect all the DSGE model parameters in the vector θ, stack the structural shocks in the vector ϵ_t, and derive a state-space representation for the $n \times 1$ vector Δy_t:

$$\Delta y_t = [\Delta \ln Y_t, \Delta \ln C_t, \Delta \ln I_t, \ln L_t, \Delta \ln(W_t/P_t), \pi_t, R_t]',$$

where Δ denotes the temporal difference operator.

3 DSGE-VARs as Tools for Model Evaluation

In addition to the DSGE model we consider a vector autoregressive specification for Δy_t. VARs are widely employed in empirical macroeconomics and often serve as benchmarks for the evaluation of dynamic equilibrium economies. We borrow from the indirect inference literature, e.g., Gourieroux, Monfort, and Renault (1993) and Smith (1993), and use the VAR as an approximating model for the DSGE model. We construct a mapping from the DSGE model parameters to the VAR parameters. As is well-known, the DSGE model leads to a restricted VAR approximation. We interpret deviations of the VAR parameters from the cross-coefficient restrictions as DSGE model misspecification. While most of the techniques described below are also applicable if the DSGE model is solved with nonlinear techniques, we use a log-linear approximation in our empirical analysis.

Broadly speaking, the goal of our analysis is to obtain estimates of the DSGE model and the VAR parameters, assess the magnitude of the DSGE model misspecification, and to learn from the discrepancy between restricted and unrestricted impulse response dynamics how to improve the specification of the DSGE model. The analysis is conducted in a Bayesian framework. Starting from a prior distribution for the DSGE model parameters θ we use the mapping from θ to the VAR coefficients to obtain a prior for the VAR parameters. This prior has the property that it concentrates in a lower dimensional subspace. Since we are concerned about misspecification, we also form a prior over deviations of the VAR parameters from the DSGE model restrictions. This prior is centered at zero and its precision is scaled by a hyperparameter λ. The hyperparameter generates a continuum of models, which we call DSGE-VAR(λ), that essentially has an unrestricted VAR as one extreme (λ is near zero) and the VAR approximation of the DSGE model at the other extreme ($\lambda = \infty$). By model we mean a joint probability distribution for the data and parameters.

Markov-Chain-Monte-Carlo (MCMC) methods are used to conduct posterior inference. We interpret the marginal likelihood function of λ as an overall measure of fit and denote its peak by $\hat{\lambda}$. A large value of $\hat{\lambda}$ and a likelihood ratio of $\lambda = \hat{\lambda}$ versus $\lambda = \infty$ close to one is interpreted as evidence in favor of the DSGE model restrictions. Impulse response comparisons of DSGE-VAR(∞) and DSGE-VAR($\hat{\lambda}$) can generate insights into the sources of DSGE model misspecification. Our approach is related to recent work by Gallant and McCulloch (2004) who proposed a Bayesian framework for indirect inference. In their analysis the approximating model is mainly a device for obtaining a likelihood function in a setting where it is computationally cumbersome to evaluate the underlying structural model. In our analysis we use the approximating model mainly as a tool to relax DSGE model restrictions and obtain an empirical specification that fits well and can serve as a benchmark for impulse response comparisons.

In the remainder of this section we present the VAR approximation of the DSGE model, the specification of the prior distribution, a characterization of the posteriors and the marginal likelihood of λ, a method to construct identified impulse responses from the DSGE-VAR, and a simple analytical illustration of our method.

3.1 VAR Approximation of the DSGE Model

We use a VAR in vector error correction form as approximating model for our analysis:

$$\Delta y_t = \Phi_0 + \Phi_\beta(\beta' y_{t-1}) + \Phi_1 \Delta y_{t-1} + \ldots + \Phi_p \Delta y_{t-p} + u_t. \tag{26}$$

We assume that the vector of reduced-form innovations $u_t \sim \mathcal{N}(0, \Sigma_u)$ conditional on past information. According to the DSGE model, the technology process Z_t generates a common trend in output, consumption, investment, and real wages. We impose this common trend structure on the approximating model by including the error correction term

$$\beta' y_{t-1} = \left[\ln C_{t-1} - \ln Y_{t-1},\ \ln I_{t-1} - \ln Y_{t-1},\ \ln(W_{t-1}/P_{t-1}) - \ln Y_{t-1} \right]'$$

on the right-hand-side of (26). We denote the dimension of Δy_t by n, define the $k \times 1$ vector $x_t = [1, (\beta' y_{t-1})', \Delta y'_{t-1}, \ldots, \Delta y'_{t-p}]'$, and let $\Phi = [\Phi_0, \Phi_\beta, \Phi_1, \ldots, \Phi_p]'$. Assuming that under the DSGE model the distribution of x_t is stationary with a non-singular covariance matrix (both conditions are satisfied for the model specified in Section 2), we define the moments $\Gamma_{YY}(\theta) = I\!\!E^D_\theta[\Delta y_t \Delta y_t']$, $\Gamma_{XX}(\theta) = I\!\!E^D_\theta[x_t x_t']$, and $\Gamma_{XY}(\theta) = I\!\!E^D_\theta[x_t \Delta y_t']$ and use a population regression to obtain the mapping from DSGE model to VAR parameters:

$$\Phi^*(\theta) = \Gamma^{-1}_{XX}(\theta)\Gamma_{XY}(\theta), \quad \Sigma^*_u(\theta) = \Gamma_{YY}(\theta) - \Gamma_{YX}(\theta)\Gamma^{-1}_{XX}(\theta)\Gamma_{XY}(\theta). \tag{27}$$

Here, $\Gamma_{YX} = \Gamma'_{XY}$. We will refer to $\Phi^*(\theta)$ and $\Sigma^*_u(\theta)$ as restriction functions.

3.2 Misspecification and Bayesian Inference

If the vector autoregressive representation of Δy_t deviates from the restriction functions $\Phi^*(\theta)$ and $\Sigma^*_u(\theta)$ then the DSGE model is misspecified. A key step in our analysis is the formulation of a prior distribution for the discrepancy between Φ and $\Phi^*(\theta)$, which we denote by Φ^Δ. We use a prior whose density is decreasing in Φ^Δ, implying that large misspecifications have low probabilities. This assumption reflects the belief that the DSGE model provides a good albeit not perfect approximation of reality.

To fix ideas we will begin by (i) ignoring the dependence of Φ^* on θ and (ii) imposing that $\Sigma_u = \Sigma^*_u$. Suppose we generate a sample of λT observations from the DSGE model, collected in the matrices Y_* and X_*. Our prior for Φ^Δ has the property that its density is proportional to the expected likelihood ratio of $\Phi^* + \Phi^\Delta$ versus Φ^*. The log-likelihood ratio is given by

$$\begin{aligned} &\ln\left[\frac{\mathcal{L}(\Phi^* + \Phi^\Delta, \Sigma^*_u | Y_*, X_*)}{\mathcal{L}(\Phi^*, \Sigma^*_u | Y_*, X_*)}\right] \\ &= -\frac{1}{2} tr\left[\Sigma^{*-1}_u\left(\Phi^{\Delta'} X_*' X_* \Phi^\Delta + 2\Phi^{*'} X_*' X_* \Phi^\Delta - 2(\Phi^* + \Phi^\Delta)' X_*' Y_* + 2\Phi^{*'} X_*' Y_*\right)\right]. \end{aligned} \tag{28}$$

Y_* denotes the $\lambda T \times n$ matrix with rows $y^{*'}_t$ and X_* is the $\lambda T \times k$ matrix with rows $x^{*'}_t$. Taking expectations under the distribution generated by the DSGE model yields

$$I\!\!E^D_\theta\left[\ln\left[\frac{\mathcal{L}(\Phi^* + \Phi^\Delta, \Sigma^*_u | Y_*, X_*)}{\mathcal{L}(\Phi^*, \Sigma^*_u | Y_*, X_*)}\right]\right] = -\frac{1}{2} tr\left[\Sigma^{*-1}_u\left(\Phi^{\Delta'} \lambda T \Gamma_{XX} \Phi^\Delta\right)\right]. \tag{29}$$

We now choose a prior density that is proportional ($\propto$) to the expected likelihood ratio:

$$p(\Phi^\Delta | \Sigma^*_u) \propto \exp\left\{-\frac{1}{2} tr\left[\lambda T \Sigma^{*-1}_u\left(\Phi^{\Delta'} \Gamma_{XX} \Phi^\Delta\right)\right]\right\}. \tag{30}$$

As the sample size λT increases the prior places more mass on misspecification matrices that are close to zero. A graphical illustration is provided in Figure 1.

In the empirical application we allow for uncertainty about θ by specifying a prior with density $p(\theta)$ and we take potential misspecification of the covariance matrix $\Sigma_u^*(\theta)$ into account. T will correspond to the size of the actual sample and λ is a hyperparameter that controls the expected magnitude of the deviations from the DSGE model restrictions. Conditional on θ our prior for the VAR coefficients takes the form

$$\begin{aligned} \Sigma_u|\theta &\sim \mathcal{IW}\left(\lambda T\Sigma_u^*(\theta), \lambda T - k\right) \\ \Phi|\Sigma_u, \theta &\sim \mathcal{N}\left(\Phi^*(\theta), \frac{1}{\lambda T}\left[\Sigma_u^{-1} \otimes \Gamma_{XX}(\theta)\right]^{-1}\right), \end{aligned} \tag{31}$$

where $\mathcal{IW}$ denotes the inverted Wishart distribution. This prior distribution is proper, i.e., has mass one, provided that $\lambda T \geq k + n$. Hence, we restrict the domain of λ to the interval $[(k+n)/T, \infty]$. The prior is identical to the one used in Del Negro and Schorfheide (2004) but its motivation is different. Del Negro and Schorfheide (2004) focused on the improvement of VARs and emphasized mixed estimation based on artificial data from a DSGE model and actual data. The present paper asks the opposite question: how can we relax DSGE model restrictions and evaluate the extent of their misspecification?

3.3 Posterior Distributions

The posterior density is proportional to the product of prior density and likelihood function. We factorize the posterior into the conditional density of the VAR parameters given the DSGE model parameters and the marginal density of the DSGE model parameters:

$$p_\lambda(\Phi, \Sigma_u, \theta|Y) = p_\lambda(\Phi, \Sigma_u|Y, \theta)p_\lambda(\theta|Y). \tag{32}$$

The actual observations are collected in the matrices Y and X and the λ-subscript indicates the dependence of the posterior on the hyperparameter. We use $\hat{\Gamma}_{XX}$, $\hat{\Gamma}_{XY}$, and $\hat{\Gamma}_{XX}$ to denote the sample autocovariances such as $\frac{1}{T}\sum x_t x_t'$. It is straightforward to show, e.g., Zellner (1971), that the posterior distribution of Φ and Σ is also of the Inverted Wishart – Normal form:

$$\begin{aligned} \Sigma_u|Y, \theta &\sim \mathcal{IW}\left(T(\lambda+1)\hat{\Sigma}_{u,b}(\theta),\ T(\lambda+1) - k\right) \\ \Phi|Y, \Sigma_u, \theta &\sim \mathcal{N}\left(\hat{\Phi}_b(\theta),\ \Sigma_u \otimes [T(\lambda\Gamma_{XX}(\theta) + \hat{\Gamma}_{XX})]^{-1}\right), \end{aligned} \tag{33}$$

where $\hat{\Phi}_b(\theta)$ and $\hat{\Sigma}_{u,b}(\theta)$ are the given by

$$\begin{aligned}\hat{\Phi}_b(\theta) &= (\lambda\Gamma_{XX}(\theta) + \hat{\Gamma}_{XX})^{-1}(\lambda\Gamma_{XY}(\theta) + \hat{\Gamma}_{XY}) \\ \hat{\Sigma}_{u,b}(\theta) &= \frac{1}{(\lambda+1)}\Big[(\lambda\Gamma_{YY}(\theta) + \hat{\Gamma}_{YY}) - (\lambda\Gamma_{YX}(\theta) + \hat{\Gamma}_{YX}) \\ &\quad \times \Big(\lambda\Gamma_{XX}(\theta) + \hat{\Gamma}_{XX}\Big)^{-1}(\lambda\Gamma_{XY}(\theta) + \hat{\Gamma}_{XY})\Big].\end{aligned}$$

Thus, the larger the weight λ of the prior, the closer the posterior mean of the VAR parameters is to $\Phi^*(\theta)$ and $\Sigma_u^*(\theta)$, the values that respect the cross-equation restrictions of the DSGE model. On the other hand, if λ equals the lower bound $(n+k)/T$ then the posterior mean is close to the OLS estimate $\hat{\Gamma}_{XX}^{-1}\hat{\Gamma}_{XY}$. The formula for the marginal posterior density of θ and the description of a MCMC algorithm that generates draws from the joint posterior of Φ, Σ_u, and θ are provided in Del Negro and Schorfheide (2004). They also demonstrate (Proposition 2) that under certain conditions the estimate of θ can be interpreted as minimum distance estimate that is obtained by projecting the VAR coefficient estimates back onto the restriction functions $\Phi^*(\theta)$ and $\Sigma_u^*(\theta)$.

3.4 The Marginal Likelihood Function of λ

We will study the fit of the DSGE model by examining the marginal likelihood function of the hyperparameter λ, which is defined as[6]

$$p(Y|\lambda) = \int p(Y|\theta,\Sigma,\Phi)p_\lambda(\theta,\Sigma,\Phi)d(\theta,\Sigma,\Phi). \tag{34}$$

For computational reasons, we only consider a finite set of values $\Lambda = \{l_1, \ldots, l_q\}$, where $l_1 = (n+k)/T$ and $l_q = \infty$. If we assign equal prior probabilities to the elements of Λ, the posterior probabilities for the hyperparameter are proportional to the marginal likelihood. Hence, we will also refer to $p(Y|\lambda)$ as the posterior of λ and denote its mode by

$$\hat{\lambda} = \text{argmax}_{\lambda\in\Lambda}\, p(Y|\lambda). \tag{35}$$

It is common in the literature, e.g., Smets and Wouters (2003) to use marginal data densities to document the fit of DSGE models relative to VARs with diffuse priors. In our framework this approach corresponds (approximately) to comparing $p(Y|\lambda)$ for the extreme values of λ, that is, $\lambda = \infty$ (DSGE model) and $\lambda = (k+n)/T$ (VAR with nearly flat prior). It is preferable to report the entire marginal likelihood function $p(Y|\lambda)$ rather than just its endpoints. The function $p(Y|\lambda)$ summarizes the time series evidence on model misspecification

[6]We use Geweke's (1999a) modified harmonic mean estimator to obtain a numerical approximation of the marginal likelihood function based on the output of the MCMC computations.

and documents by how much the restrictions of the DSGE model have to be relaxed to balance in-sample fit and model complexity.

To illustrate the properties of the marginal likelihood function $p(Y|\lambda)$ it is instructive to consider the following univariate example. Suppose the VAR takes the special form of an AR(1) model:

$$y_t = \phi y_{t-1} + u_t, \quad u_t \sim iid\mathcal{N}(0,1) \tag{36}$$

and the DSGE model restricts ϕ to be equal to ϕ^*. We will denote the DSGE model implied autocovariances of order 0 and 1 by γ_0 and γ_1, respectively. Moreover, $\hat{\gamma}_0$ and $\hat{\gamma}_1$ are sample autocovariances based on T observations. The prior in (31) simplifies to

$$\phi \sim \mathcal{N}\left(\phi^*, \frac{1}{\lambda T \gamma_0}\right). \tag{37}$$

For this simple model the marginal likelihood of λ takes the following form

$$\ln p(Y|\lambda,\phi^*) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\tilde{\sigma}^2(\lambda,\phi^*) - \frac{1}{2}c(\lambda,\phi^*). \tag{38}$$

The term $\tilde{\sigma}^2(\lambda,\phi^*)$ measures the in-sample one-step-ahead forecast error and can be written as

$$\tilde{\sigma}^2(\lambda,\phi^*) = \hat{\gamma}_0 + \lambda\gamma_0 - \frac{(\hat{\gamma}_1 + \lambda\gamma_1)^2}{\hat{\gamma}_0 + \lambda\gamma_0} - \lambda\left(\gamma_0 - \frac{\gamma_1^2}{\gamma_0}\right). \tag{39}$$

It can be verified that as λ approaches zero $\tilde{\sigma}^2(\lambda,\phi^*)$ converges to the OLS forecast error, whereas as $\lambda \longrightarrow \infty$ we obtain the in-sample forecast error under the restriction $\phi = \phi^*$. Formally,

$$\lim_{\lambda\longrightarrow 0}\tilde{\sigma}^2(\lambda,\phi^*) = \frac{1}{T}\sum(y_t - \hat{\phi}y_{t-1})^2, \quad \lim_{\lambda\longrightarrow\infty}\tilde{\sigma}^2(\lambda,\phi^*) = \frac{1}{T}\sum(y_t - \phi^* y_{t-1})^2,$$

where $\hat{\phi} = \hat{\gamma}_1/\hat{\gamma}_0$. Moreover, $\tilde{\sigma}^2(\lambda,\phi^*)$ is monotonically increasing in λ, that is, the larger λ the worse the in-sample fit. The third term in (38) can be interpreted as a penalty for model complexity and is of the form

$$c(\lambda,\phi^*) = \ln\left(1 + \frac{\hat{\gamma}_0}{\lambda\gamma_0}\right). \tag{40}$$

In the context of a standard regressor selection problem model complexity is tied to the number of included regressors and the penalty is an increasing function of the number of parameters that are being estimated. In our setup, model complexity is a continuous function of the hyperparameter λ. If $\lambda = \infty$ there is no parameter to estimate in the AR(1) example and the complexity, or alternatively, the dimensionality of the model is zero. If $\lambda = 0$ then the autoregressive parameter is completely unrestricted and the dimensionality is one. Accordingly, the penalty term (40) is monotonically decreasing in λ. As λ approaches zero and the prior becomes more diffuse the penalty diverges to infinity.

Several features of the marginal data density are noteworthy. First, the marginal likelihood function is either monotonically decreasing, increasing, or it has an interior maximum. If an interior maximum exists, it is given by

$$\hat{\lambda} = \frac{\gamma_0 \hat{\gamma}_0^2}{T(\hat{\gamma}_0 \gamma_1 - \gamma_0 \hat{\gamma}_1)^2 - \gamma_0^2 \hat{\gamma}_0}. \tag{41}$$

Thus, if the sample autocovariances are very different from the autocovariances derived under the restriction $\phi = \phi^*$, the marginal likelihood peaks at a small value of λ. As the discrepancy between sample and DSGE model autocovariances decreases, $\hat{\lambda}$ increases, and the marginal likelihood will eventually attain its maximum at $\hat{\lambda} = \infty$.

Second, as λ approaches zero, the marginal log likelihood function tends to minus infinity. In the context of high-dimensional VARs this feature of the marginal likelihood function enforces parsimony and prevents the use of over-parameterized specifications that cannot be precisely estimated based on the fairly short samples that are available to macroeconomists. In these cases, a naive posterior odds comparison of VAR and DSGE model based on the endpoints of the marginal likelihood function, corresponding to a VAR with diffuse prior (small λ) and a VAR with DSGE model restrictions imposed, may not be very informative because it automatically tends to favor the restricted specification. This phenomenon arises more generally in Bayesian posterior odds comparisons and is called Lindley's Paradox. Rather than limiting the attention to extremes, our procedure creates a continuum of prior distributions and evaluates the marginal likelihood function for a range of hyperparameter values. The magnitudes of $\hat{\lambda}$ and $p(Y|\lambda = \hat{\lambda}, \phi^*)/p(Y|\lambda = \infty, \phi^*)$ provide measures of overall fit of the DSGE model.

Third, consider the comparison of two models $\mathcal{M}_1$ and $\mathcal{M}_2$. In the context of our univariate example these models correspond to different restrictions, $\phi^*_{(1)}$ and $\phi^*_{(2)}$, say. In our empirical analysis we will compare the marginal likelihood functions associated with different DSGE model specifications. For small values of λ the goodness-of-fit terms $\tilde{\sigma}^2(\lambda, \phi^*_{(1)})$ and $\tilde{\sigma}^2(\lambda, \phi^*_{(2)})$ are essentially identical and differences in marginal likelihoods are due to differences in the penalty terms. For large values of λ, on the other hand, penalty differentials are less important and the marginal likelihood comparison is driven by the relative in-sample fit of the two restricted specifications. If the autocovariances associated with $\mathcal{M}_1$ are closer to the sample autocovariances than the $\mathcal{M}_2$ autocovariances, then according to (41) $\hat{\lambda}_{(1)}$ tends to be larger than $\hat{\lambda}_{(2)}$.

3.5 Impulse Response Function Comparisons

The goal of our impulse response function comparisons is to document in which dimensions the DSGE model dynamics are (in)consistent with the data. There is an extensive literature that evaluates DSGE models by comparing their impulse responses to those obtained from VARs: Cogley and Nason (1994), Rotemberg and Woodford (1997), Schorfheide (2000), Boivin and Giannoni (2005), and Christiano, Eichenbaum, and Evans (2005), to name a few. Such a comparison faces two challenges. First, for the VAR to be a meaningful benchmark it has to fit the data better, accounting for model complexity, than the DSGE model. Many authors use simple least squares techniques which lead to very noisy coefficient estimates in high-dimensional systems. The imprecise coefficient estimates translate into impulse response function estimates that in a mean-squared-error sense are worse than the estimates obtained directly from the DSGE model. Second, the VAR has to be expressed in terms of structural shocks. It is typically difficult to find identification schemes that are consistent with the DSGE model and simultaneously identify an entire vector of structural shocks in a high-dimensional VAR.

In the DSGE-VAR procedure the benchmark is given by DSGE-VAR($\hat{\lambda}$), the model that attains the highest marginal likelihood. Therefore, by construction our procedure meets the first challenge: the benchmark model attains a better fit – penalized for model complexity to avoid over-parameterization – and tends to deliver more reliable impulse response estimates than the restrictive DSGE model. The spirit of our evaluation is to keep the autocovariance sequence associated with the benchmark model as close to the DSGE model as possible without sacrificing the ability to track the historical time series. Next, we describe how the DSGE-VAR analysis can address the second challenge, identification.

While the DSGE model provides a state-space representation in terms of the vector of structural shocks ϵ_t from which we can calculate the impulse responses directly, the DSGE-VARs are specified in terms of reduced form innovations u_t. Hence, the first step toward the comparison of impulse responses is the identification of structural shocks in the DSGE-VARs. We express the one-step-ahead VAR forecast errors as a linear function of the structural shocks ϵ_t:

$$u_t = \Sigma_{tr} \Omega \epsilon_t, \tag{42}$$

where Σ_{tr} is the Cholesky decomposition of Σ and Ω is an orthonormal matrix with the property $\Omega\Omega' = I$. The matrix Ω is not identifiable from the data since the likelihood function of the VAR depends only on the covariance matrix

$$\Sigma_u = \Sigma_{tr}\Omega\Omega'\Sigma_{tr}' = \Sigma_{tr}\Sigma_{tr}'.$$

Hence, we have to make auxiliary assumptions to determine Ω.

We follow Del Negro and Schorfheide (2004) and construct Ω as follows. The state space representation of the DSGE model is identified in the sense that for each value of θ there is a unique matrix $A_0(\theta)$ that determines the contemporaneous effect of ϵ_t on Δy_t. Using a QR factorization of $A_0(\theta)$, the initial response of Δy_t to the structural shocks can be can be uniquely decomposed into

$$\left(\frac{\partial \Delta y_t}{\partial \epsilon_t'}\right)_{DSGE} = A_0(\theta) = \Sigma_{tr}^*(\theta)\Omega^*(\theta), \tag{43}$$

where $\Sigma_{tr}^*(\theta)$ is lower triangular and $\Omega^*(\theta)$ is orthonormal. According to Equation (26) the initial impact of ϵ_t on the endogenous variables Δy_t in the VAR is given by

$$\left(\frac{\partial \Delta y_t}{\partial \epsilon_t'}\right)_{VAR} = \Sigma_{tr}\Omega. \tag{44}$$

To identify the DSGE-VAR, we maintain the triangularization of its covariance matrix Σ_u and replace the rotation Ω in Equation (44) with the function $\Omega^*(\theta)$ that appears in (43).

Using the rotation matrix $\Omega^*(\theta)$, we turn the reduced-form DSGE-VAR into an identified DSGE-VAR. The prior distribution

$$p_\lambda(\theta, \Phi, \Sigma_u) = p(\theta)p_\lambda(\Phi, \Sigma_u|\theta)$$

together with the mapping $\Omega = \Omega^*(\theta)$ induces a prior distribution for the coefficients of the structural VAR, which is then updated using the likelihood function of the reduced-form VAR. Since beliefs about the VAR parameters are centered around the restriction functions $\Phi^*(\theta)$ and $\Sigma_u^*(\theta)$ our prior implies, roughly speaking, that beliefs about impulse responses to structural shocks are centered around the DSGE model responses, even for small values of the hyperparameter λ. However, the smaller λ, the wider are the probability intervals for the response functions. Our approach differs from much of the empirical literature on identified VARs as it ties identification closely to the underlying DSGE model. We do not view this feature as a shortcoming. Since the premise of our analysis is that the DSGE model provides a good albeit not perfect approximation of reality, strong views about the identification of particular structural shocks can and should be directly incorporated into the underlying DSGE model.

Two pairwise comparisons of impulse responses are interesting: (i) DSGE model versus DSGE-VAR(∞) and (ii) DSGE-VAR(∞) versus DSGE-VAR($\hat{\lambda}$). In our application we are working with a log-linearized DSGE model that can be expressed a vector autoregressive moving average (VARMA). The first comparison provides some insights about the accuracy of the VAR approximation, while the second comparison helps to understand the dimensions in which the DSGE model is misspecified. If the DSGE model's MA polynomial is non-invertible or has roots near the unit circle, then the approximation by a finite-order VAR could be poor.[7] If, on the other hand, the MA polynomial is well approximated by a few autoregressive terms, then our identification procedure for the DSGE-VAR is able to recover the DSGE model responses associated with the VARMA representation. In our application we find that for parameter values of θ near the posterior mode the discrepancy between DSGE and DSGE-VAR(∞) responses is fairly small, in particular in the short-run. However, as in the indirect inference literature, our analysis remains coherent and insightful even if the VAR provides only an approximation to the underlying DSGE model.

A comparison of DSGE-VAR($\hat{\lambda}$) and DSGE-VAR(∞) responses illustrates the discrepancy between the coefficient estimates that optimally relax the DSGE model restrictions and the restricted estimates. If the posterior estimates of the VAR parameters are close to the restriction functions $\Phi^*(\theta)$ and $\Sigma_u^*(\theta)$ then the DSGE-VAR($\hat{\lambda}$) and DSGE-VAR(∞) will be very similar. If on the other hand, the posterior estimates strongly deviate from the restriction function, the discrepancy between the impulse responses potentially provides valuable insights on how to improve the underlying DSGE model.

4 The Data

All data are obtained from Haver Analytics (Haver mnemonics are in italics). Real output, consumption of nondurables and services, and investment (defined as gross private domestic investment plus consumption of durables) are obtained by dividing the nominal series (*GDP*, *C* - *CD*, and *I* + *CD*, respectively) by population 16 years and older (*LN16N*), and deflating using the chained-price GDP deflator (*JGDP*). The real wage is computed by dividing Compensation of Employees (*YCOMP*) by total hours worked and the GDP deflator. Note that compensation per hours includes wages as well as employer contribution. It accounts for

[7] Fernandez-Villaverde, Rubio-Ramirez, and Sargent (2004) provide necessary and sufficient conditions for the invertibility of the moving average components of linear state-space models. We did not replace the VAR specification in (26) by an unrestricted VARMA to avoid the approximation error for two reasons. First, VARs have established themselves as popular and powerful tools for empirical research and forecasting in macroeconomics. Second, from a computational perspective the posterior of DSGE-VAR is much easier to analyze than the posterior of a DSGE-VARMA.

both wage and salary workers and proprietors. Our measure of hours worked is computed by taking total hours worked reported in the National Income and Product Accounts (NIPA), which is at annual frequency, and interpolating it using growth rates computed from hours of all persons in the non-farm business sector (*LXNFH*). We divide hours worked by *LN16N* to convert them into per capita terms. Our broad measure of hours worked is consistent with our definition of both wages and output in the economy. All growth rates are computed using quarter-to-quarter log differences and then multiplied by 100 to convert them into annualized percentages. Inflation rates are defined as log differences of the GDP deflator and converted into annualized percentages. The nominal rate corresponds to the effective Federal Funds Rate (*FFED*), also in percent. Data are available from QIII:1954 to QI:2004.

5 Empirical Results

The empirical analysis is conducted in four parts. The first part reports on the prior and posterior distributions for the DSGE model parameters. The second part discusses the evidence of misspecification in the New Keynesian model. We calculate marginal likelihood functions for the hyperparameter λ and study the discrepancy in the impulse responses to monetary and technology shocks between the DSGE-VAR($\hat{\lambda}$) and the DSGE-VAR(∞). In the third part, we use the DSGE-VAR framework for the comparison of different DSGE model specifications. We strip the baseline model of some of its frictions (habit formation and price/wage indexation) and ask to what extent the time series fit suffers as a consequence. Finally, we report some results on pseudo-out-of-sample forecasting accuracy.

Unless otherwise noted, all results are based on thirty years of observations ($T = 120$), starting in QII:1974 and ending in QI:2004. The same sample size is used in the pseudo-out-of-sample forecasting exercise. Beginning from QIII:1954 we construct 58 rolling samples of 120 observations, estimate the DSGE-VARs as well as the state-space representation of the DSGE model for each sample, and compute forecast error statistics. All MCMC results are based on 110,000 draws from the relevant posterior distribution, discarding the first 10,000. We checked whether 110,000 draws were sufficient by repeating the MCMC computations from over-dispersed starting points, verifying that we obtain the same results for parameter estimates and log marginal likelihood functions.

The lag-length p of the DSGE-VAR is 4. To make the DSGE-VAR estimates comparable to the estimates of the state-space representation of the DSGE model, we used in both cases likelihood functions that condition on the four observations that are needed to initialize lags in period $t = 1$ as well as on the cointegration vector $\beta' y_0$. Since DSGE-VAR(∞) is not equivalent to the state-space representation of the DSGE model we will adopt the convention that whenever we refer to the estimation of the DSGE model, we mean its state-space representation.

5.1 Priors for the DSGE Parameters

Priors for the DSGE model parameters are provided in the first four columns of Table 1. All intervals reported in the text are 90% probability intervals. The priors for the degree of price and wage stickiness, ζ_p and ζ_w, are both centered at 0.6, which implies that firms and households re-optimize their prices and wages on average every two and half quarters. The 90% interval is very wide and encompasses findings in micro-level studies of price adjustments such as Bils and Klenow (2004). The priors for the degree of price and wage indexation, ι_p and ι_w, are nearly uniform over the unit interval. The prior for the adjustment cost parameter s'' is taken from Smets and Wouters (2003) and is consistent with the values that Christiano, Eichenbaum, and Evans (2005) use when matching DSGE impulse response functions to consumption and investment, among other variables, to VAR responses.

Our prior for the habit persistence parameter h is centered at 0.7, which is the value used by Boldrin, Christiano, and Fisher (2001). These authors find that $h = 0.7$ enhances the ability of a standard DSGE model to account for key asset market statistics. The prior for a' implies that in response to a 1% increase in the return to capital, utilization rates rise by 0.1 to 0.3%. These numbers are considerably smaller than the one used by Christiano, Eichenbaum, and Evans (2005). The 90% interval for the prior distribution on ν_l implies that the Frisch labor supply elasticity lies between 0.3 and 1.3, reflecting the micro-level estimates at the lower end, and the estimates of Kimball and Shapiro (2003) and Chang and Kim (2006) at the upper end.

We use a pre-sample of observations from QI:1960 to QI:1974 to choose the prior means for the parameters that determine steady states. The prior mean for the technology growth rate is 2% per year. The annualized steady state inflation rate lies between 0.5 and 5.5% and the prior for the inverse of the discount factor r^* implies a growth adjusted real interest rate of 4% on average. The prior means for the capital share α, the substitution parameter λ_f, and the steady state government share $1 - 1/g$ are chosen to capture the labor share of 0.57, the investment-to-output ratio of 0.24, and the government share of 0.21 in the pre-sample. The distribution for ψ_1 and ψ_2 is approximately centered at Taylor's (1993) values, whereas the smoothing parameter lies in the range from 0.18 to 0.83.

Since we model the level of technology Z_t as a unit root root process, the prior for ρ_z, which measures the serial correlation of technology growth z_t, is centered at 0.2. The priors for ρ_μ (shocks to the capital accumulation equation), and ρ_g (government spending) are quite tight around 0.8 in order to prevent these parameters from hitting the boundary. The priors for the remaining autocorrelation coefficients of the structural shocks – ρ_φ (preferences of leisure), ρ_b (overall preference shifter), ρ_{λ_f} (price markup shocks)– are fairly diffuse and centered around 0.6. Finally, the priors for the standard deviation parameters are chosen to obtain realistic magnitudes for the implied volatility of the endogenous variables. Throughout the analysis we fix the capital depreciation rate $\delta = 0.025$ and $\lambda_w = 0.3$. The parameter λ_w affects the substitution elasticity between different types of labor. Unlike λ_f it is not identifiable from the steady-state relationships. We introduce a parameter L_{adj} that captures the units of measured hours worked. In our model, we choose φ such that in steady state each household supplies one unit of labor. A prior for L_{adj} is chosen based on quarterly per capita hours worked in the pre-sample.

5.2 Posteriors for the DSGE Parameters

The remaining columns of Table 1 report on the posterior estimates of the DSGE model parameters, for both the DSGE model and the estimation of the DSGE-VAR($\hat{\lambda}$). As described later in detail, for the sample beginning in QII:1974 the value of $\hat{\lambda}$ is 1.25. We start by focusing on the parameter estimates for the state space representation of the DSGE model. The comparison of the 90% coverage intervals suggests that likelihood contains information about most of the parameters. Three exceptions are the parameters $a^{'}$, ν_l, and ρ_z, for which prior and posterior intervals roughly overlap. The parameter estimates for the DSGE model are also generally in line with those of Smets and Wouters (2005), which is not surprising since our model specification and choice of prior is similar to theirs. In particular, the model displays a relatively high degree of price and wage stickiness, as measured by the probability that firms (wage setters) cannot change their price (wage) in a given period. The posterior means of ζ_p and ζ_w are 0.83 and 0.89, respectively. The estimated degree of indexation is about 0.7 for both prices and wages. For some of the structural shocks, notably ϕ_t and $\lambda_{f,t}$, the degree of persistence is not as high as that found in Smets and Wouters (2005).

We now turn to the parameter estimates obtained from the DSGE-VAR($\hat{\lambda}$). Del Negro and Schorfheide (2004) showed that as the prior on the VAR parameters becomes more diffuse information about the DSGE model parameters accumulates more slowly. In the limit, when $\lambda = 0$ the DSGE-VAR(λ) likelihood contains no information about the parameter vector θ and the posterior will be identical to the prior. Hence, in general we expect that for $\hat{\lambda} < \infty$ the DSGE-VAR($\hat{\lambda}$) posteriors to be closer to the prior than the DSGE model posterior. Table 1 confirms that for many of the parameters, including the degree of price and wage stickiness, the policy parameters, and some of the autocorrelation coefficients, the DSGE-VAR($\hat{\lambda}$) estimates indeed lie between the DSGE posterior and the prior distribution. One exception are the standard deviations of the structural shocks, which are estimated to be lower under DSGE-VAR($\hat{\lambda}$) than under the DSGE model regardless of the prior.

5.3 Evidence of Misspecification in the New Keynesian Model

Smets and Wouters (2003, Table 2) found for Euro area data that a large-scale new-Keynesian DSGE models can attain a larger marginal likelihood than VARs with training sample prior and specific versions of the Minnesota prior. This result has had a considerable impact on applied macroeconomists and policymakers, as it suggests that New Keynesian DSGE models have achieved a degree of sophistication that makes them competitive with more densely parameterized models such as VARs. In this subsection we revisit Smets and Wouters' findings using the DSGE-VAR procedure. We make three distinct points based on marginal likelihood functions and impulse response comparisons. First, the posterior odds of a DSGE model versus a VAR with a fairly diffuse prior do not provide a particularly robust assessment of fit. Small changes in the sample period can lead to reversals of the model ranking. The DSGE-VAR analysis, on the other hand, is much less sensitive to changes in the sample period. Second, there is strong evidence of misspecification in the New Keynesian model, suggesting that forecasts and policy recommendations obtained from this class of models should be viewed with some degree of skepticism. Finally, on the positive side we find that accounting for misspecification by optimally relaxing the DSGE model restrictions does not alter the responses to a monetary policy shock in any significant way, both qualitatively and quantitatively. Thus, in spite of its deficiencies, the New Keynesian DSGE model can indeed generate realistic predictions of the effects of unanticipated changes in monetary policy.

5.3.1 The Marginal Likelihood Function of λ

The two panels of Figure 2 show the logarithm of the marginal likelihood of DSGE-VAR(λ) for different values of λ, as well as for the DSGE model. The values of λ considered are 0.33 (smallest λ value for which we have a proper prior), 0.5, 0.75, 1, 1.25, 1.5, 2, 5, and ∞. We re-scale the x-axis according to $x = \lambda/(1+\lambda)$. In the top panel of Figure 2 depicts the marginal likelihood function for the the 30-year sample beginning in QII:1974, which is the sample used for most of the subsequent analysis. The bottom panel is based on 30-year sample that begins four years earlier in QII:1970.

The comparison between the two extremes – the VAR with loose prior on the left-hand side of the plot and the DSGE model on the right-hand side – leads to opposite conclusions depending on the sample period. In the QII:1974 – QI:2004 sample the difference in log-marginal likelihoods between the DSGE model and DSGE-VAR(0.33) is 5, which translates into posterior odds that are roughly 150 to 1 in favor of the DSGE model. Conversely, for the QII:1970 – QI:2000 sample the difference is -14, overwhelmingly against the DSGE model. This result confirms Sims' (2003) conjecture that marginal likelihood comparisons among "far-apart" models are not robust. The four years of difference between the two samples are very unlikely to contain major shifts in the economy, and therefore should not cause a change in the DSGE model's assessment.

The lack of robustness in the comparison between the two extremes contrasts with the robustness of the overall shape of the marginal likelihood function. In both panels, this function has an inverted U-shape. The marginal likelihood increases sharply as λ moves from 0.33 to 0.75, is roughly flat for values between 0.75 and 1.25, and subsequently decreases, first gradually and then more rapidly, as λ exceeds 1.5. The substantial drop in marginal likelihood between DSGE-VAR($\hat{\lambda}$) and DSGE-VAR(∞) is strong evidence of misspecification for the New-Keynesian model: As the prior tightly concentrates in the neighborhood of the cross-equation restrictions imposed by the DSGE model, the in-sample fit of the DSGE-VAR deteriorates. Del Negro and Schorfheide (2006) show that the shape of the posterior distribution of λ is roughly the same for all the 58 30-year rolling samples considered in the forecasting exercise in Section 5.4. The evidence of misspecification for the New-Keynesian model is therefore robust to the choice of the sample.

This inverted-U shape with peaks between 0.75 and 1.25 contrasts with the pattern we would expect if the data were generated by the DSGE model. The AR(1) example in Section 3.4 suggests that if the sample autocovariances were close to the population autocovariances implied by the DSGE model, then the marginal likelihood function would peak at a much larger value of λ and possibly be monotonically increasing. This is confirmed by simulation results reported in An and Schorfheide (2006), who generate observations from a small-scale DSGE model and then calculate marginal likelihood functions for λ which are indeed monotone in λ.

5.3.2 Impulse Response Function Comparisons

In order to gain further insights about the misspecification of the DSGE model we proceed by comparing impulse responses from the DSGE-VAR(∞) to our benchmark specification DSGE-VAR($\hat{\lambda}$). It turns out that in our application the approximation error of the DSGE-VAR(∞) relative to the state-space representation of the DSGE model is small (see Appendix). Consequently, the impulse responses from the DSGE-VAR(∞), in particular to a technology and a monetary policy shock, are very similar to those from the DSGE model.

We subsequently focus on the impulse response functions that have received the most attention in the literature: responses to monetary policy and technology shocks. The full set of 49 response functions can be found in the Appendix. Figure 3 depicts mean responses to one-standard deviation shocks for the DSGE-VAR(∞) (gray solid lines), the DSGE-VAR($\hat{\lambda}$) (dark dash-and-dotted lines), and 90% bands (dark dotted lines) for DSGE-VAR($\hat{\lambda}$).[8] The top panels of Figure 3 show that the impulse response functions with respect to a monetary policy shock for DSGE-VAR(∞) match those for DSGE-VAR($\hat{\lambda}$), not only qualitatively but also – by and large – quantitatively. Both in the DSGE-VAR(∞) and in the DSGE-VAR($\hat{\lambda}$) output, consumption, investment and hours display a hump-shaped response to the policy shock, although quantitatively the hump for investment is more pronounced in the data than it is in the DSGE model. Unlike in Christiano, Eichenbaum, and Evans (2005), our DSGE model implies that monetary policy shocks are observed contemporaneously. Yet, thanks to various sources of inertia, including habit formation, the initial impact of the shock on real variables is very small. The response of inflation is the only dimension where DSGE model and data disagree: according to the DSGE-VAR($\hat{\lambda}$) it is more sluggish than in the DSGE model. In summary, as in Christiano, Eichenbaum, and Evans (2005) we find that the DSGE model's impulse response to a policy shock are in agreement with the data.

[8]The responses are computed based on the respective posterior draws for the DSGE-VAR(∞) and DSGE-VAR($\hat{\lambda}$). Alternatively, one could evaluate for each θ draw from the DSGE-VAR($\hat{\lambda}$) posterior the restriction functions $\Phi^*(\theta)$ and $\Sigma_u^*(\theta)$ and calculate the corresponding impulse responses to to illustrate how far the DSGE-VAR($\hat{\lambda}$) estimates are away from the restriction functions.

This finding may not be too surprising, given that this specific model was written with this purpose in mind. Yet it is comforting to know that it is robust to a different data set and identification procedure.

The bottom panels of Figure 3 show that the responses to a technology shock have similar shapes for the DSGE-VAR(∞) and DSGE-VAR($\hat{\lambda}$), but they appear to be quantitatively different. The technology shock seems to have a larger effect in the DSGE-VAR(∞). The amplification is due to a larger estimate of the shock standard deviation caused by poorer in-sample fit of the DSGE-VAR(∞) relative to the DSGE-VAR($\hat{\lambda}$). The differences between the response functions disappear if the technology shocks in the two models are re-normalized to have the same long-run effect on output.[9]

In conclusion, we find that the DSGE model's misspecification does not translate in impulse responses to monetary policy or technology shocks that are very different between the DSGE model and the benchmark DSGE-VAR($\hat{\lambda}$). Many macroeconomists believe that these two shocks provide a very important source of business cycle fluctuations. Our results suggests that business cycle research has to a large extent been successful in developing a model that can procedure realistic responses to these shocks. However, a non-negligible fraction of fluctuations is attributed to the remaining five shocks in the model. We document in the Appendix that for some of the shocks, such as μ_t, which affects the shadow price of installed capital, DSGE-VAR(∞) and DSGE-VAR($\hat{\lambda}$) differ substantially, in particular in the long-run, suggesting that some low-frequency implications of the model are at odds with the data.

5.4 Comparing DSGE Model Specifications

The DSGE model used in this paper is rich in terms of nominal and real frictions. An important part of the empirical analysis in Smets and Wouters (2003) and Christiano, Eichen-

[9]According to Altig, Christiano, Eichenbaum, and Lindé's (2004) analysis inflation in the DSGE model does essentially not move in response to a permanent technology shock. We find that it does. Moreover, the inflation response is consistent with our benchmark impulse response function obtained from the DSGE-VAR($\hat{\lambda}$). We conjecture this difference is due to the estimation procedure used: Altig *et al.* estimate their DSGE model by matching impulse response functions. Technology shocks in their VAR are identified through long-run restrictions which tend to be imprecisely estimated. Hence, when minimizing the discrepancy between VAR and DSGE responses, more weight is placed on the responses to the monetary shocks. But as the top panel Figure 3 shows, in the data inflation react with a delay to the monetary shock. Therefore a sluggish response of inflation is wired into their estimates, and translates into a sluggish response to a technology shock as well. Our likelihood-based estimation implicitly places more weight on reproducing the response of inflation to a technology shock.

baum, and Evans (2005) is to assess which of these frictions are important to fit the data. Smets and Wouters (2003) use marginal likelihood comparisons, eliminating one friction at a time and computing posterior odds relative to the baseline specification. Christiano, Eichenbaum, and Evans (2005) study whether the impulse responses of a model without a specific friction can match the VAR's impulse responses as well as the baseline model.

In this paper we use DSGE-VARs to assess the importance of two particular features of the DSGE model: price and wage indexation and habit formation. We will refer to the model without wage and price indexation as the *No Indexation* model and to the model without habit formation as the *No Habit* model, while the standard DSGE model used up to now will be referred to as the *Baseline* model. We will document that habit formation is important to fit the data, whereas the evidence in favor of indexation is weak.

We compare the marginal likelihood of λ for the baseline model with that of the two alternative specifications. Our example in Section 3.4 suggests that as the mismatch between sample autocovariances and population autocovariances implied by the DSGE model increases, $\hat{\lambda}$ decreases and the marginal likelihood function shifts downward. Therefore, we can infer from the magnitude of the south-west shift in the marginal likelihood function the extent to which a specific friction is useful in fitting the data.

We emphasized previously that in the absence of a more elaborate DSGE model a comparison of impulse responses between the DSGE-VAR(∞) and DSGE-VAR($\hat{\lambda}$) can generate important insights on how to improve the model specification. Using the hindsight from our analysis of the *Baseline* model, we will subsequently examine whether such a comparison for the *No Indexation* and *No Habit* models reveal in what directions these models need to be augmented.

5.4.1 Evidence from the Marginal Likelihood Functions

Figure 4 resembles the top panel of Figure 2, except that we overlay the marginal likelihood functions for the *Baseline* (solid line), the *No Indexation* (dashed line), and the *No Habit* (dash-and-dotted line) model. Smets and Wouters (2003) dogmatically enforced the cross-equation restrictions of the DSGE model specifications, which leads to a comparison of the three marginal likelihood values on the right edge of Figure 4. Both alternative specifications are strongly rejected in favor of the *Baseline*, even though the rejection for the *No Indexation* is not as stark as for the *No Habit* model.

The evidence contained in the overall posterior distribution of λ against the *No Habit* model is equally strong. Figure 4 shows that relative to the *Baseline* model the marginal likelihood of λ not only shifts down, but also to the left. Translating the marginal likelihood values into posterior probabilities, for the *No Habit* model there is very little probability mass associated with values of λ greater than one. Conversely, the left-ward shift for the *No Indexation* model is much less pronounced and the marginal likelihood remains fairly flat for values of λ between 0.75 and 2.

5.4.2 Evidence from Impulse Response Functions

Suppose all we have available is the *No Habit* (*No Indexation*) model. Can we learn from the impulse response comparison between the DSGE-VAR(∞) and DSGE-VAR($\hat{\lambda}$) that some important feature is missing from the structural model? Figure 5 depicts the mean impulse responses to monetary policy (top panel) and technology shocks (bottom panel) for DSGE-VAR(∞) (gray solid line) and DSGE-VAR($\hat{\lambda}$) (dark dash-and-dotted lines), as well a the 90% bands (dark dotted lines) for DSGE-VAR($\hat{\lambda}$). Figure 5 is obtained based on the *No Habit* model. Therefore the benchmark DSGE-VAR($\hat{\lambda}$) in Figure 5 differs from that in Figure 3 for two reasons. First, the value of $\hat{\lambda}$ is lower, as can be appreciated from Figure 4. Second, the prior for the VAR coefficients is based on the *No Habit* model as opposed to the *Baseline* model.

A comparison of Figures 5 and 3 indicates that the initial responses to a monetary policy shock of output, consumption, and hours for the *No Habit* DSGE model look very different from those of the *Baseline* DSGE model. All real variables with the exception of investment and real wages now display a strong initial reaction to the monetary shock, which contrasts with the hump-shaped responses in the DSGE-VAR($\hat{\lambda}$). Even if Figure 3 were not available to the researcher, the comparison between the impulse responses for $\lambda = \infty$ and $\lambda = \hat{\lambda}$ in Figure 5 would reveal that something is amiss in DSGE model without habit formation. A similar analysis applies to the responses to a technology shock (bottom panel of Figure 5), where consumption reacts strongly on impact according to DSGE-VAR(∞), compared to the more gradual response in the DSGE-VAR($\hat{\lambda}$). Importantly, the benchmark responses in Figures 5 and 3 are similar, both qualitatively and quantitatively, in spite of the fact that the underlying set of cross-equation restrictions is different. Thus, even under *No Habit* the DSGE-VAR($\hat{\lambda}$) provides a reasonable benchmark although the DSGE model misspecification is seemingly stronger than for the *Baseline* model.

Figure 6 shows the impulse responses for the *No Indexation* model. Unlike Figure 5, Figure 6 shows no stark divergence between DSGE-VAR(∞) and the benchmark, DSGE-VAR($\hat{\lambda}$). Indeed the impulse response functions in both panels of Figure 6 are quite similar to those of Figures 3. The change in the cross-equation restrictions does not seem to translate into an appreciable change in the transmission mechanism of monetary policy and technology shocks. Perhaps the main difference consists in the response of inflation to technology shocks, which is somewhat hump-shaped in Figure 3 but not in Figure 6. Quantitatively however this difference does not amount to much, as the hump is small.

In conclusion the evidence from the DSGE-VAR procedure against the *No Indexation* model is not nearly as strong as that against the *No Habit* model. These findings suggest that habit persistence in preferences substantially improves the fit of the DSGE model. Hence, those who believe that habit persistence is not a "structural" feature may have to introduce alternative mechanisms that deliver similar effects. Simply eliminating habit persistence comes at a cost in terms of fit. On the contrary, the evidence in favor of price and wage indexation is not nearly as strong, in spite of the fact that the marginal likelihood comparison between DSGE models (Figure 4) – if taken literally – rejects the *No Indexation* model in favor of the *Baseline*.

5.5 Pseudo-Out-of-Sample Forecast Accuracy

We now discuss the pseudo-out-sample fit of DSGE-VAR(∞) and compare it to that of the DSGE-VAR($\hat{\lambda}$) and an unrestricted VAR. The out-of-sample forecasting accuracy is assessed based on a rolling sample starting in QIV:1985 and ending in QI:2000, for a total of 58 periods. At each date of the rolling sample we use the previous 120 observations to re-estimate the models, and the following eight quarters to assess forecasting accuracy, which is measured by the root mean squared error (RMSE) of the forecast. For the variables that enter the VAR in growth rates (output, consumption, investment, real wage) and inflation we forecast cumulative changes. For instance, the RMSE of inflation for eight quarters ahead forecasts measures the error in forecasting cumulative inflation over the next two years (in essence, average inflation), as opposed to quarter-to-quarter inflation in two years. The DSGE-VARs are re-estimated for each of the 58 samples. As discussed above, the value of $\hat{\lambda}$ hovers between 0.75 and 1.25.

Table 2 documents for each series and forecast horizon the root mean square error (RMSE) of the unrestricted VAR, as well as the percentage improvement in forecasting accuracy (whenever positive) of DSGE-VAR($\hat{\lambda}$) and DSGE-VAR(∞) relative to the VAR. The last three rows of the Table report the corresponding figures for the multivariate statistic, a summary measure of joint forecasting performance, which is computed as the converse of the log-determinant of the variance-covariance matrix of forecast errors.

Table 2 shows that for the multivariate statistic, and for most variables, DSGE-VAR($\hat{\lambda}$) improves over the VAR for all forecasting horizons. Short-run consumption forecasts and long-run investment forecasts are an exception. Interestingly, there seems to be a trade-off between forecasting consumption and investment. This trade-off reflects the fact that all three models considered in Table 2 are error correction models with the same long-run cointegrating restrictions on output, consumption, investment, and the real wage. These cointegrating restrictions are at odds with the data. Hence, accurate forecasts for some of these variables result in inaccurate forecasts for others, given that not all series grow proportionally in the long run as the model predicts. Another manifestation of this phenomenon is the fact that DSGE-VAR(∞) outperforms the other two models in forecasting the real wage in the long run, but performs very poorly in forecasting both output and investment. In summary, the fact that the DSGE model imposes these long-run cointegrating restrictions results in a serious limitation of its forecasting ability. To the extent that DSGE-VAR inherits the same long-run restrictions, its accuracy suffers as well.

For the remaining variables, DSGE-VAR($\hat{\lambda}$) is roughly as accurate as the unrestricted VAR in terms of hours per capita, while DSGE-VAR (∞) is far worse, especially in the long run. Conversely, DSGE-VAR (∞) performs well in terms of the nominal variables, inflation and the interest rate. For inflation DSGE-VAR (∞)'s forecasting accuracy is inferior to that of DSGE-VAR($\hat{\lambda}$), but far better than that of the unrestricted VAR. For the nominal interest rate, DSGE-VAR(∞) outperforms DSGE-VAR($\hat{\lambda}$) for longer forecast horizons, while in the short run the two models have roughly the same forecasting performance.

Extending the analysis of Section 5.4, we now discuss the comparison of the out-of-sample forecasting performance across models. Figure 7 shows the one-quarter ahead percentage improvement in the multivariate forecast statistic relative to the unrestricted VAR for the *Baseline* (solid line), *No Indexation* (dashed line), and the *No Habit* (dash-and-dotted line) models, as a function of λ. Note that the benchmark used for the computation of the percentage improvement – the unrestricted VAR – is the same for all three models. Figure 7 focuses on one-period-ahead forecasting accuracy to facilitate the comparison with the results in Figure 4, which were based on the marginal likelihood.

The results in Figure 7 agree in a number of dimensions with those in Figure 4. The inverted-U shape that characterized the posterior distribution of λ for each of the model in Figure 4 also describes the improvement in forecasting accuracy relative to the VAR. Results documented in Del Negro and Schorfheide (2006) show that this inverted-U shape characterizes the improvement in forecasting accuracy for all forecasting horizons from one to eight quarters ahead. Relaxing, yet not ignoring the cross-equation restrictions leads to an improvement in fit and forecasting performance. Consistently with the overall message from the previous section, the *No Indexation* and the *Baseline* model perform roughly as well in terms of multivariate statistic, while the forecasting accuracy worsens considerably for the *No Habit* model relative to the *Baseline* model as the DSGE prior becomes too tight.

6 Conclusions

Smets and Wouters (2003) showed that large-scale New-Keynesian models with real and nominal rigidities can fit as well as VARs estimated under diffuse priors, and possibly better. This result implies that these models are becoming a tool usable for quantitative analysis by policy making institutions. In addition, it implies that vector autoregressions estimated with simple least squares techniques, or from a Bayesian perspective, estimated under a very diffuse prior, many not provide a reliable benchmark. In turn, this suggests that more elaborate tools for model evaluation are necessary. Using techniques developed in Del Negro and Schorfheide (2004) we constructed a reliable benchmark by systematically relaxing the restrictions that the DSGE model poses on a vector autoregressive to optimize its fit measured by the marginal likelihood function. We argued that comparing the DSGE model's and the benchmark's impulse response function can shed light on the nature of the DSGE model's misspecification.

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Table 1: DSGE Model's Parameter Estimates

	Prior				DSGE-VECM($\hat{\lambda}$) Post.		DSGE Post.	
	Distr.	P(1)	P(2)	Interval	Mean	Interval	Mean	Interval
α	$\mathcal{B}$	0.33	0.10	[0.16 , 0.49]	0.23	[0.20 , 0.26]	0.26	[0.23 , 0.29]
ζ_p	$\mathcal{B}$	0.60	0.20	[0.29 , 0.93]	0.79	[0.72 , 0.86]	0.83	[0.79 , 0.87]
ι_p	$\mathcal{B}$	0.50	0.28	[0.08 , 0.95]	0.75	[0.53 , 1.00]	0.76	[0.57 , 0.97]
s''	$\mathcal{G}$	4.00	1.50	[1.60 , 6.28]	4.57	[2.60 , 6.61]	5.70	[3.34 , 7.90]
h	$\mathcal{B}$	0.70	0.05	[0.62 , 0.78]	0.75	[0.70 , 0.81]	0.81	[0.77 , 0.85]
a'	$\mathcal{G}$	0.20	0.10	[0.05 , 0.35]	0.27	[0.10 , 0.43]	0.19	[0.07 , 0.32]
ν_l	$\mathcal{G}$	2.00	0.75	[0.81 , 3.15]	1.69	[0.66 , 2.74]	2.09	[0.95 , 3.19]
ζ_w	$\mathcal{B}$	0.60	0.20	[0.29 , 0.94]	0.79	[0.70 , 0.87]	0.89	[0.84 , 0.93]
ι_w	$\mathcal{B}$	0.50	0.28	[0.05 , 0.93]	0.45	[0.04 , 0.80]	0.70	[0.47 , 0.96]
r^*	$\mathcal{G}$	2.00	1.00	[0.49 , 3.49]	1.36	[0.41 , 2.28]	1.52	[0.48 , 2.50]
ψ_1	$\mathcal{G}$	1.50	0.40	[0.99 , 2.09]	1.80	[1.42 , 2.19]	2.21	[1.79 , 2.63]
ψ_2	$\mathcal{G}$	0.20	0.10	[0.05 , 0.35]	0.16	[0.09 , 0.22]	0.07	[0.03 , 0.10]
ρ_r	$\mathcal{B}$	0.50	0.20	[0.18 , 0.83]	0.76	[0.70 , 0.83]	0.82	[0.78 , 0.86]
π^*	$\mathcal{N}$	3.01	1.50	[0.56 , 5.46]	2.98	[0.89 , 5.19]	5.98	[4.61 , 7.38]
γ	$\mathcal{G}$	2.00	1.00	[0.46 , 3.47]	1.08	[0.39 , 1.80]	0.94	[0.40 , 1.43]
λ_f	$\mathcal{G}$	0.15	0.10	[0.01 , 0.29]	0.35	[0.29 , 0.42]	0.29	[0.24 , 0.34]
g^*	$\mathcal{G}$	0.30	0.10	[0.14 , 0.46]	0.19	[0.13 , 0.24]	0.23	[0.20 , 0.26]
L^{adj}	$\mathcal{N}$	252.0	10.0	[235.5 , 268.4]	257.6	[244.3 , 271.5]	245.2	[233.5 , 255.3]

Notes: See Section 2 for a definition of the DSGE model's parameters, and Section 4 for a description of the data. $\mathcal{B}$ is Beta, $\mathcal{G}$ is Gamma, $\mathcal{IG}$ is Inverse Gamma, and $\mathcal{N}$ is Normal distribution. $P(1)$ and $P(2)$ denote means and standard deviations for $\mathcal{B}$, $\mathcal{G}$, and $\mathcal{N}$ distributions; s and ν for the $\mathcal{IG}$ distribution, where $p_{\mathcal{IG}}(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$. The effective prior is truncated at the boundary of the determinacy region and the prior probability interval reflects this truncation. All probability intervals are 90% credible. The following parameters are fixed: $\delta = 0.025$, $\lambda_w = 0.3$, $\Phi = 0$. Estimation results are based on the sample period QII:1974 - QI:2004.

Table 1: (continued)

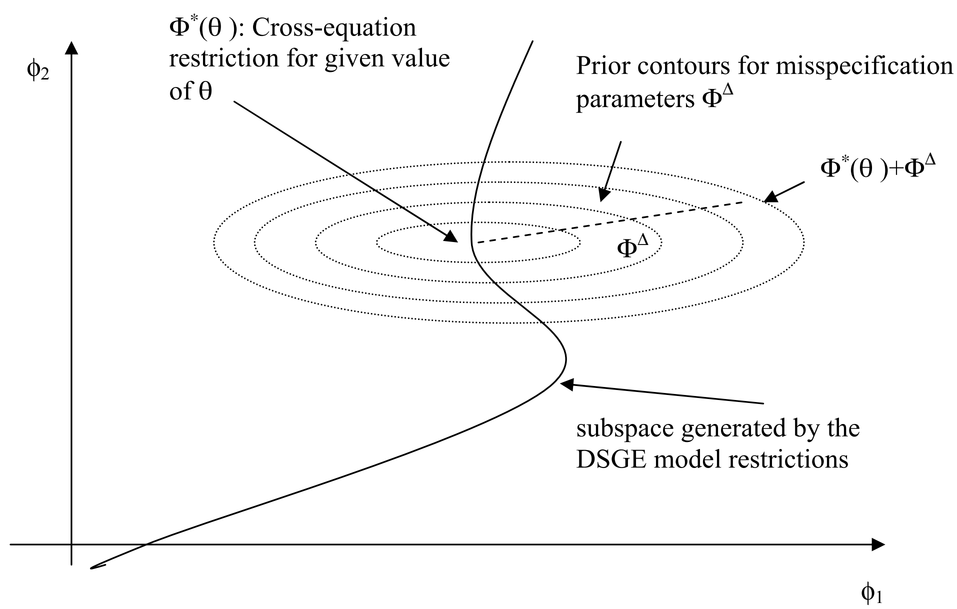
	Prior				DSGE-VECM($\hat{\lambda}$) Post.		DSGE Post.	
	Distr.	P(1)	P(2)	Interval	Mean	Interval	Mean	Interval
ρ_z	$\mathcal{B}$	0.20	0.10	[0.04 , 0.35]	0.20	[0.08 , 0.32]	0.20	[0.09 , 0.31]
ρ_ϕ	$\mathcal{B}$	0.60	0.20	[0.29 , 0.93]	0.38	[0.20 , 0.58]	0.25	[0.11 , 0.37]
ρ_{λ_f}	$\mathcal{B}$	0.60	0.20	[0.28 , 0.93]	0.11	[0.03 , 0.21]	0.12	[0.02 , 0.21]
ρ_μ	$\mathcal{B}$	0.80	0.05	[0.72 , 0.88]	0.74	[0.68 , 0.81]	0.87	[0.81 , 0.94]
ρ_b	$\mathcal{B}$	0.60	0.20	[0.29 , 0.93]	0.80	[0.68 , 0.92]	0.92	[0.86 , 0.97]
ρ_g	$\mathcal{B}$	0.80	0.05	[0.72 , 0.88]	0.90	[0.85 , 0.96]	0.95	[0.93 , 0.97]
σ_z	$\mathcal{IG}$	1.33	2.64	[0.31 , 2.34]	0.57	[0.48 , 0.65]	0.82	[0.72 , 0.91]
σ_ϕ	$\mathcal{IG}$	7.12	11.82	[1.64 , 12.57]	11.83	[4.41 , 19.84]	40.54	[18.21 , 64.09]
σ_{λ_f}	$\mathcal{IG}$	1.33	2.36	[0.31 , 2.34]	0.21	[0.18 , 0.25]	0.24	[0.21 , 0.28]
σ_μ	$\mathcal{IG}$	1.32	2.29	[0.30 , 2.33]	0.55	[0.43 , 0.67]	0.66	[0.54 , 0.78]
σ_b	$\mathcal{IG}$	1.33	2.32	[0.30 , 2.33]	0.32	[0.24 , 0.41]	0.54	[0.36 , 0.71]
σ_g	$\mathcal{IG}$	1.34	5.60	[0.31 , 2.34]	0.30	[0.26 , 0.34]	0.38	[0.34 , 0.42]
σ_r	$\mathcal{IG}$	0.36	0.85	[0.08 , 0.62]	0.18	[0.15 , 0.21]	0.28	[0.25 , 0.31]

Table 2: Pseudo-Out-of-Sample RMSEs: Percentage Improvement relative to VAR

			forecast horizon				
			1	2	4	6	8
Y	DSGE-VAR($\hat{\lambda}$),		16.3	14.1	12.5	13.5	13.6
	DSGE-VAR(∞),		0.9	-17.6	-56.5	-82.5	-102.9
	VAR,	RMSE:	*0.67*	*0.97*	*1.68*	*2.38*	*2.98*
C	DSGE-VAR($\hat{\lambda}$),		-6.8	-7.6	7.1	16.6	21.5
	DSGE-VAR(∞),		-15.7	-21.4	-0.8	11.3	12.0
	VAR,	RMSE:	*0.42*	*0.62*	*1.06*	*1.56*	*2.03*
I	DSGE-VAR($\hat{\lambda}$),		17.8	8.0	-5.0	-11.5	-17.2
	DSGE-VAR(∞),		-4.2	-41.2	-101.0	-135.3	-157.8
	VAR,	RMSE:	*2.67*	*3.98*	*6.59*	*9.14*	*11.45*
H	DSGE-VAR($\hat{\lambda}$),		10.0	10.9	-0.6	-0.0	0.7
	DSGE-VAR(∞),		-13.6	-37.9	-95.4	-116.5	-127.2
	VAR,	RMSE:	*0.58*	*0.92*	*1.56*	*2.26*	*2.88*
W	DSGE-VAR($\hat{\lambda}$),		8.2	11.7	11.1	14.9	18.4
	DSGE-VAR(∞),		6.7	12.7	18.1	27.0	36.6
	VAR,	RMSE:	*0.65*	*1.06*	*1.72*	*2.28*	*2.82*
Inflation	DSGE-VAR($\hat{\lambda}$),		10.7	10.9	22.9	31.0	36.6
	DSGE-VAR(∞),		8.4	4.2	10.4	21.1	29.6
	VAR,	RMSE:	*0.25*	*0.47*	*0.98*	*1.68*	*2.42*
R	DSGE-VAR($\hat{\lambda}$),		27.3	23.4	9.2	7.0	9.1
	DSGE-VAR(∞),		27.7	17.8	3.2	8.2	17.1
	VAR,	RMSE:	*0.68*	*1.14*	*1.63*	*2.11*	*2.64*
Multivariate	DSGE-VAR($\hat{\lambda}$),		11.0	8.8	6.1	9.4	9.4
Statistic	DSGE-VAR(∞),		3.8	-2.1	-6.9	-2.7	-0.2
	VAR,	RMSE:	*0.68*	*0.23*	*-0.18*	*-0.47*	*-0.65*

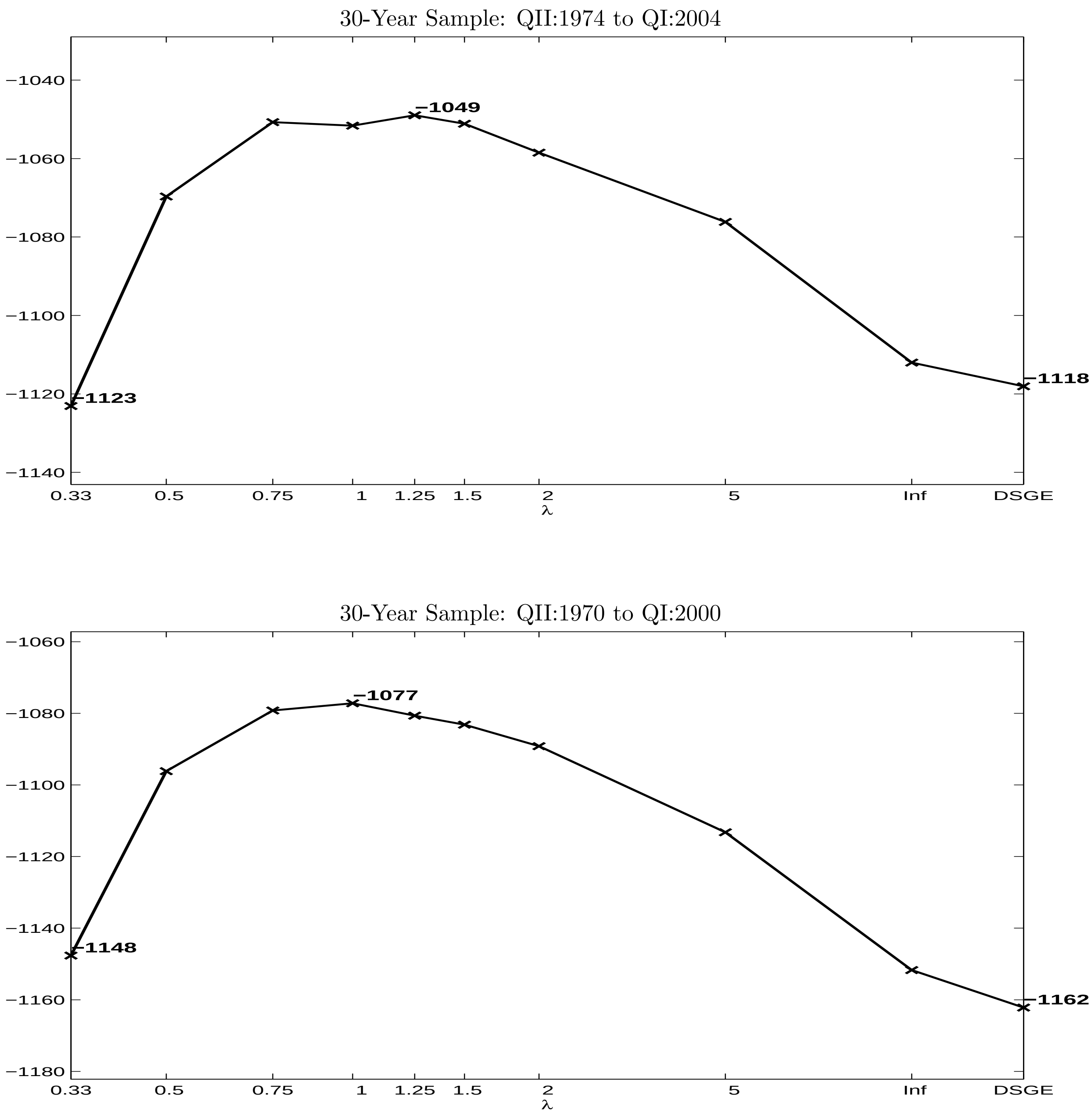
Notes: Results are based on 58 rolling samples of 120 observations. For each rolling sample, we estimate DSGE model and DSGE-VARs, compute $\hat{\lambda}$, and calculate pseudo-out-of-sample forecast errors for the subsequent 8 periods. For each variable, the table reports RMSE of the forecast from the VAR and improvements in forecast accuracy obtained by the DSGE model and the DSGE-VAR($\hat{\lambda}$). Improvements (positive entries) are measured by the percentage reduction in RMSE. The multivariate statistic is computed as the converse of the log-determinant of the variance-covariance matrix of forecast errors. The forecast horizon is measured in quarters. See Section 4 for a description of the data.

Figure 1: Stylized View of DSGE Model Misspecification



Notes: $\Phi = [\phi_1, \phi_2]'$ can be interpreted as the VAR parameters, and $\Phi^*(\theta)$ is the restriction function implied by the DSGE model.

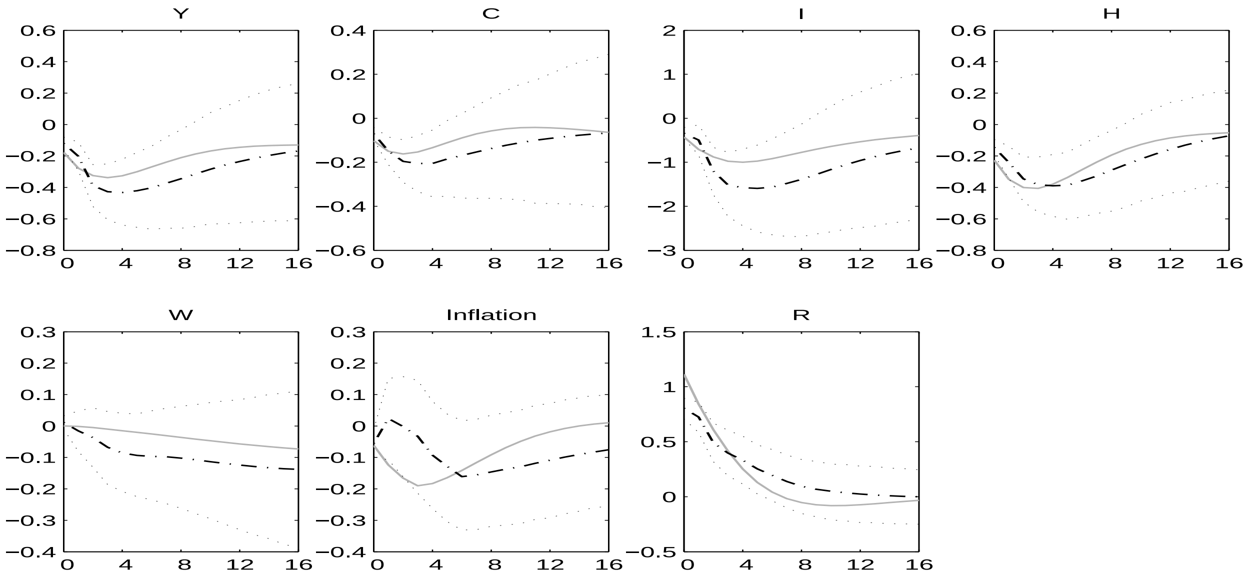
Figure 2: Marginal Likelihood as a Function of λ



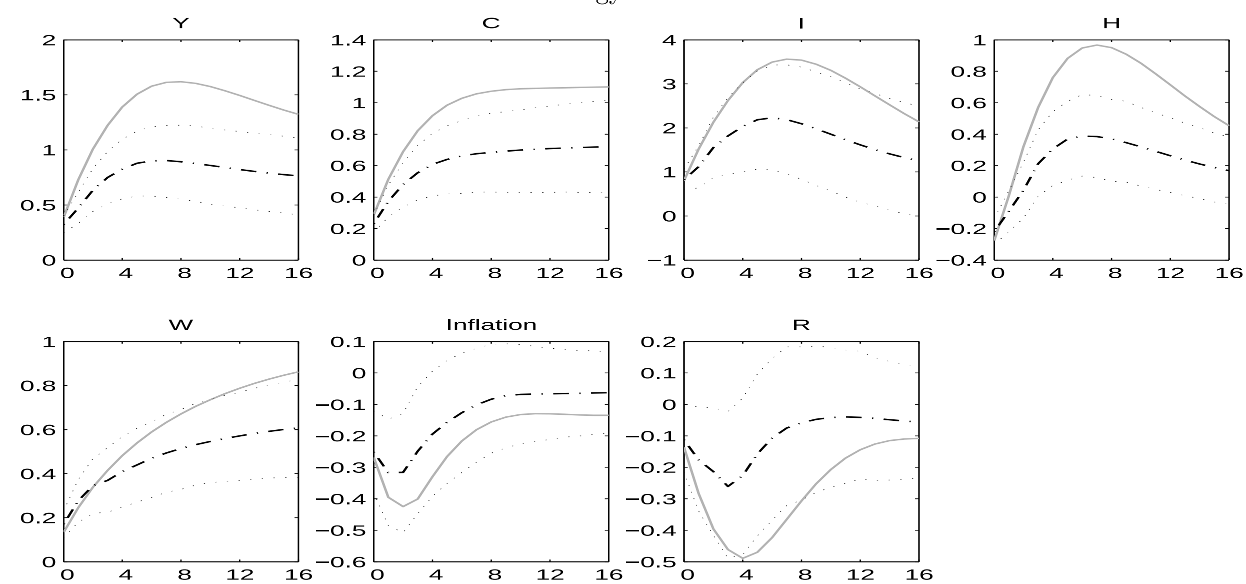
Notes: The two panels depict the log marginal likelihood function on the y-axis and the corresponding value of λ, re-scaled between via the transformation $\lambda/(1+\lambda)$, on the x-axis. The right endpoint depicts the log marginal likelihood for the state-space representation of the DSGE model.

Figure 3: IMPULSE RESPONSE FUNCTIONS: DSGE-VAR($\hat{\lambda}$) VS. DSGE-VAR(∞)

Monetary Policy Shocks

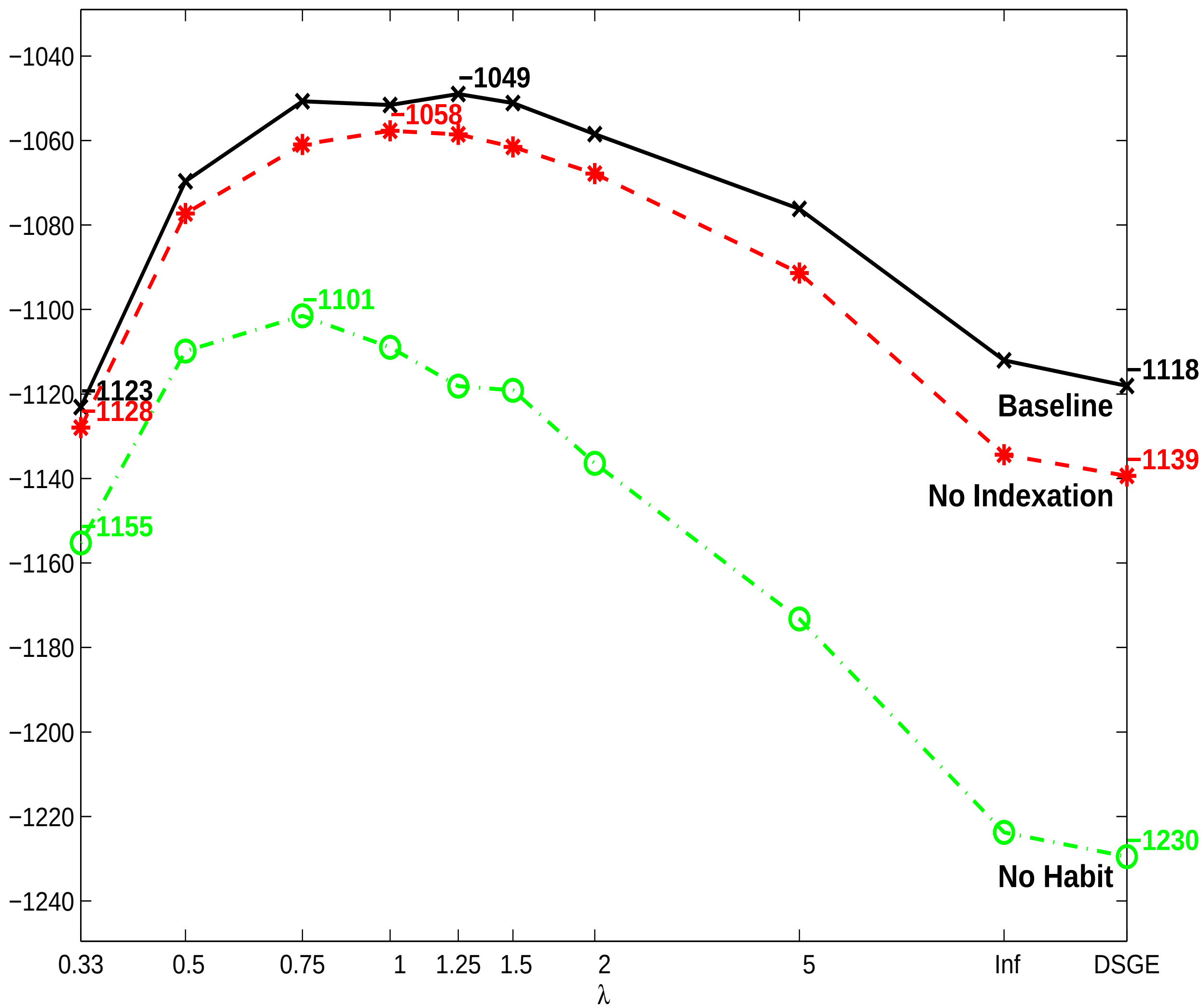


Technology Shocks



Notes: Figure depicts posterior mean responses for the DSGE-VAR(∞) (gray solid lines) and the DSGE-VAR($\hat{\lambda}$) (dark dash-and-dotted lines), and 90% bands (dark dotted lines) for DSGE-VAR($\hat{\lambda}$).

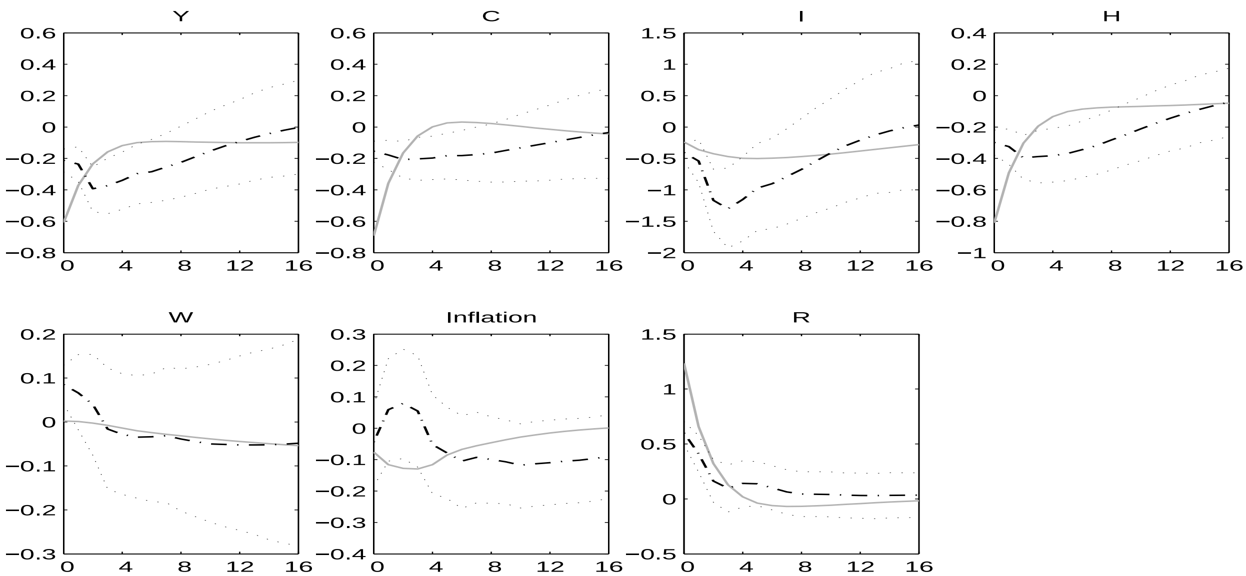
Figure 4: Marginal Likelihood as a Function of λ: Comparison across Models



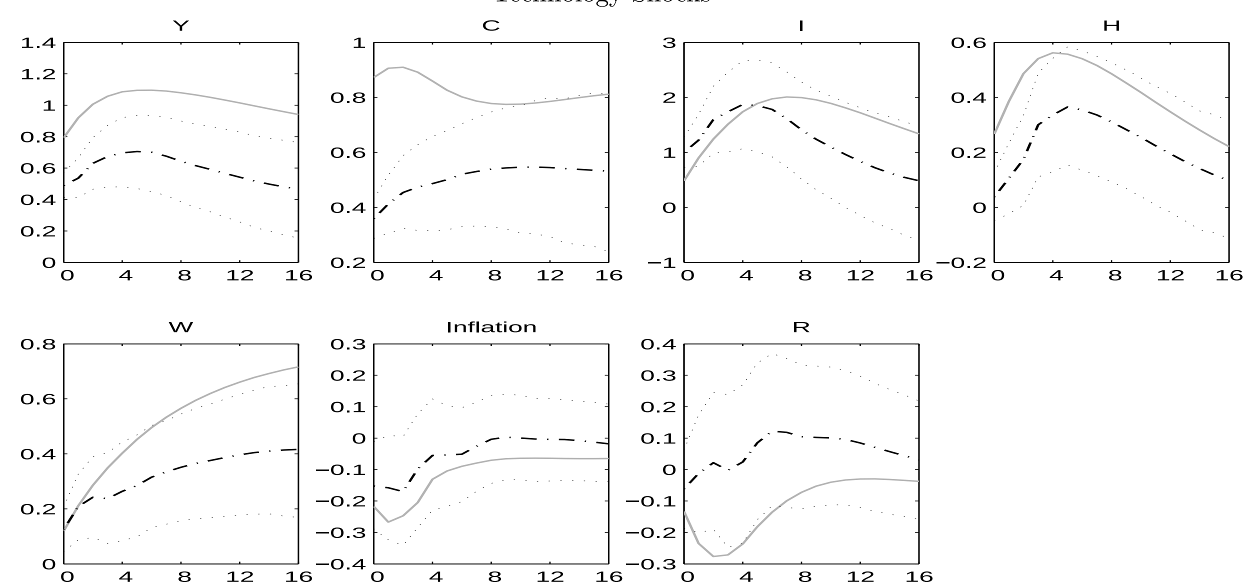
Notes: See Figure 2

Figure 5: Impulse Response Functions for the *No Habit* Model: DSGE-VAR($\hat{\lambda}$) vs. DSGE-VAR(∞)

Monetary Policy Shocks



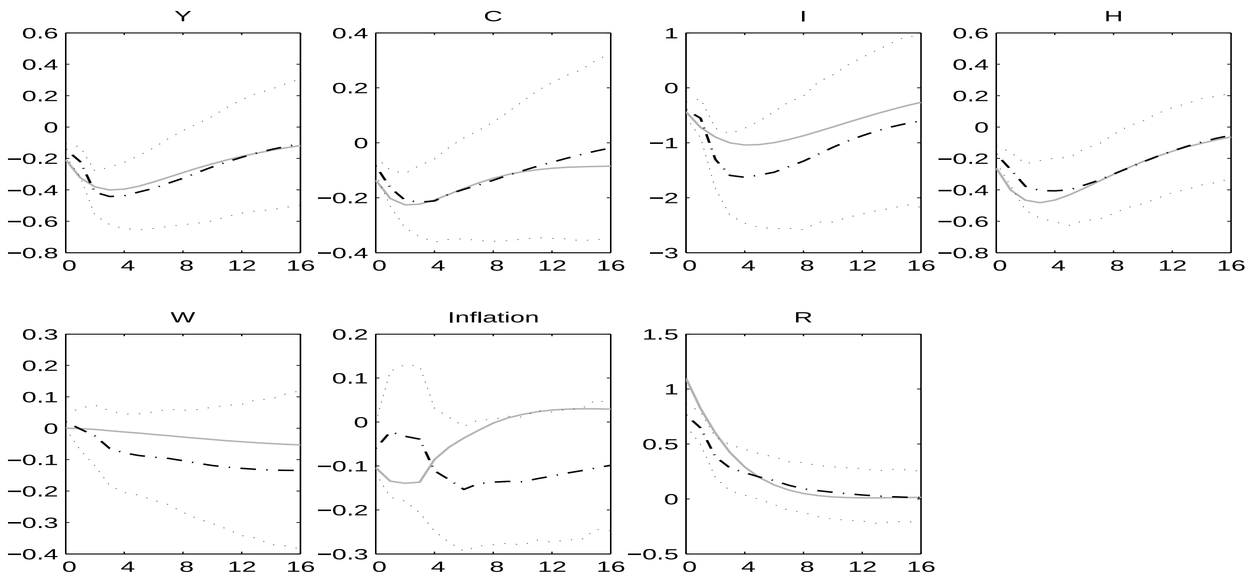
Technology Shocks



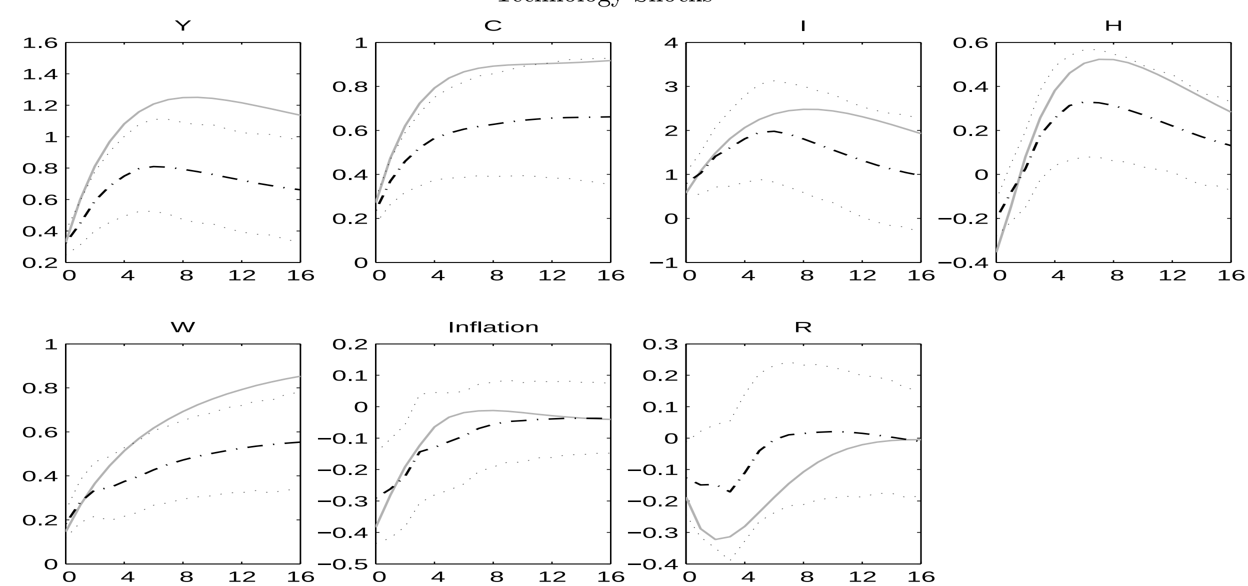
Notes: See Figure 3

Figure 6: IMPULSE RESPONSE FUNCTIONS FOR THE *No Indexation* MODEL: DSGE-VAR($\hat{\lambda}$) VS. DSGE-VAR(∞)

Monetary Policy Shocks

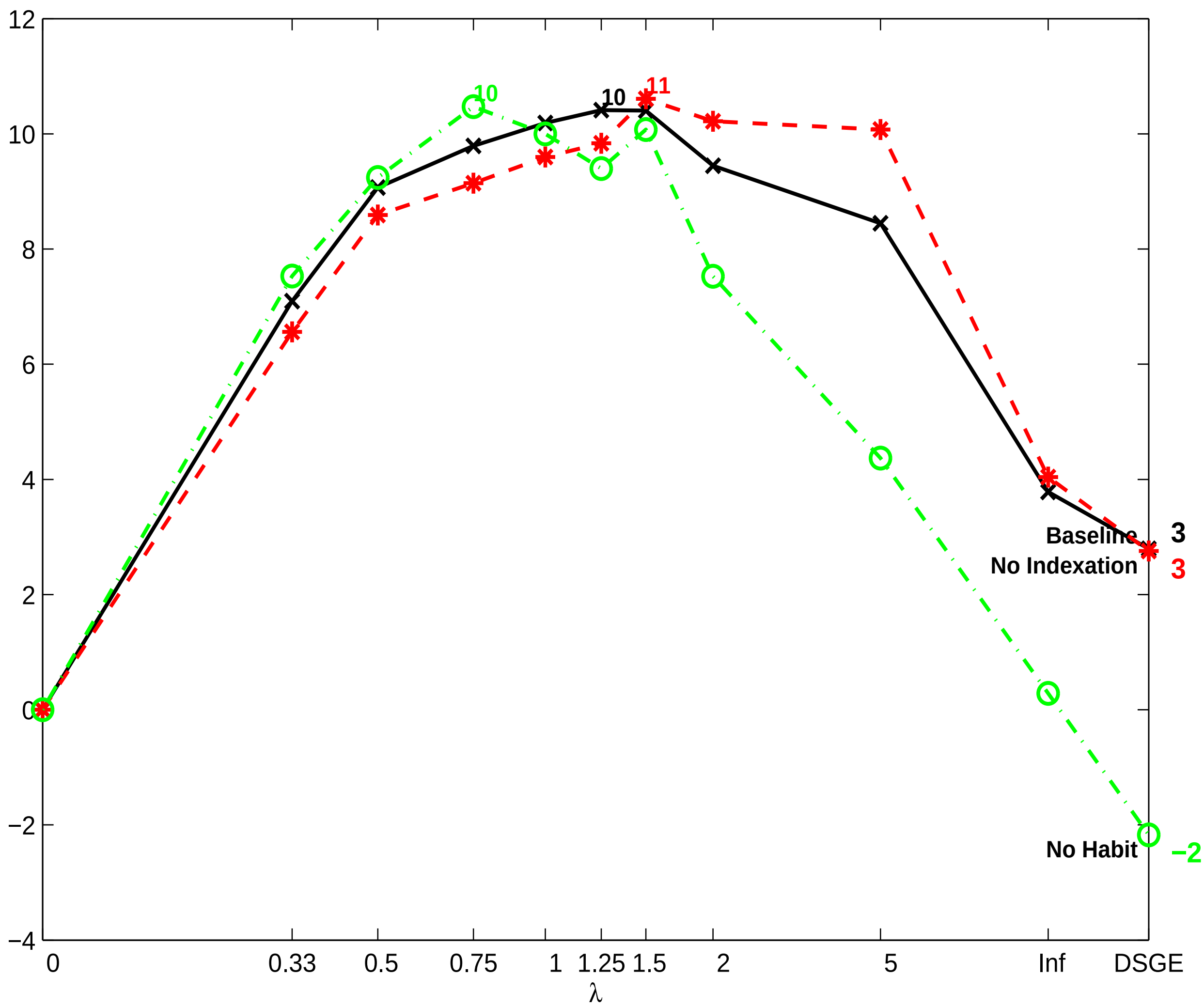


Technology Shocks



Notes: See Figure 3

Figure 7: One-period Ahead Root Mean Square Error Summary: Model Comparison



Notes: Figure depicts asymptotic risks as a function of local misspecification: *solid* is $\lambda = \hat{\lambda}$, dashed is $\lambda = \infty$, and dotted is $\lambda = T^{-1}$.

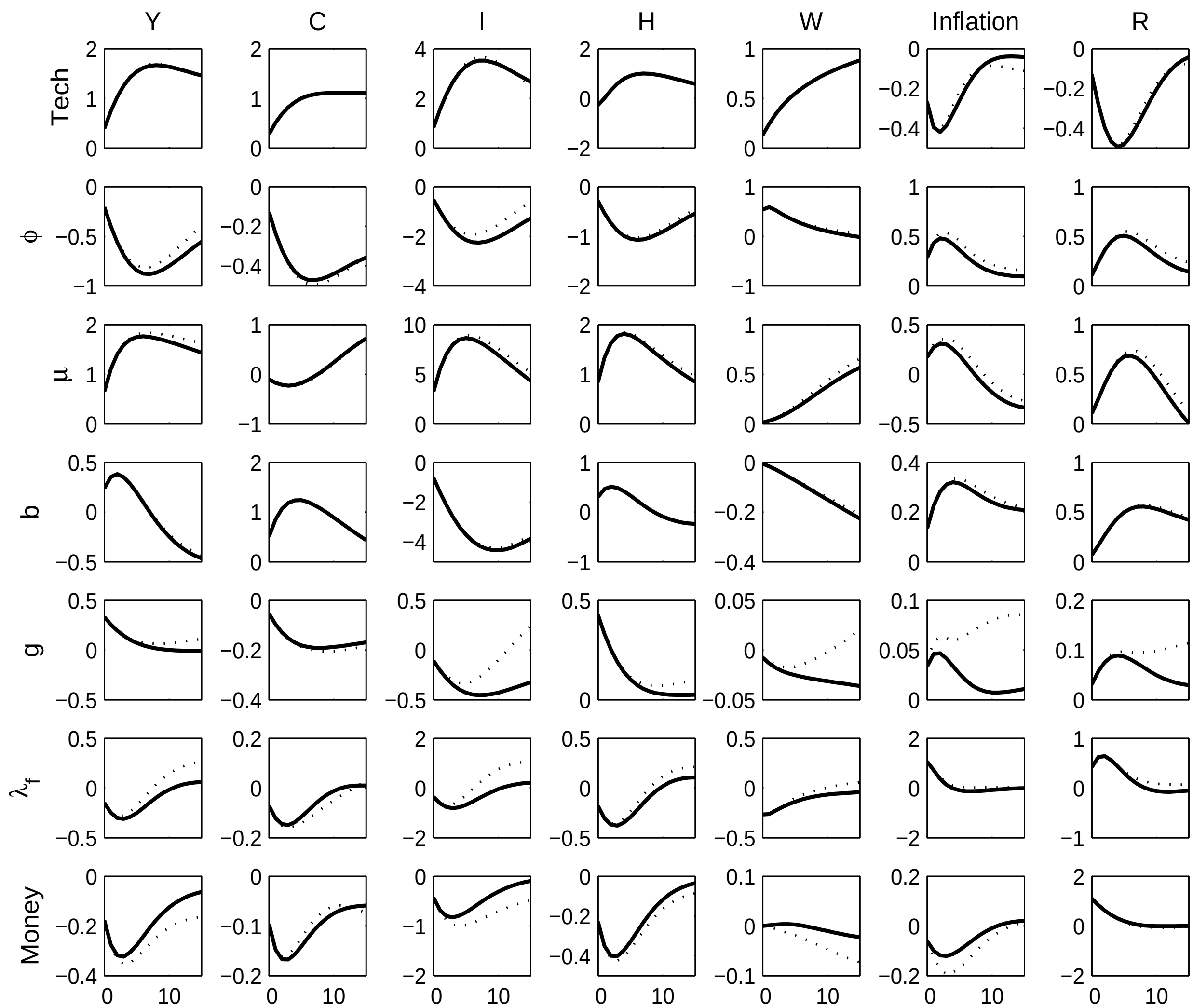
A Appendix

Figure A-1 shows the impulse responses of the endogenous variables to one-standard deviation shocks for the DSGE-VAR(∞) (dotted lines) and for the state-space representation of the DSGE model (solid lines). Both impulse responses are computed using the same set of DSGE model parameters, namely the mean estimates for the DSGE model reported in Table 1.

Figure A-2 depicts mean responses of the endogenous variables to one-standard deviation shocks for the DSGE-VAR(∞) (gray solid lines), the DSGE-VAR($\hat{\lambda}$) (dark dash-and-dotted lines), and 90% bands (dark dotted lines) for DSGE-VAR($\hat{\lambda}$).

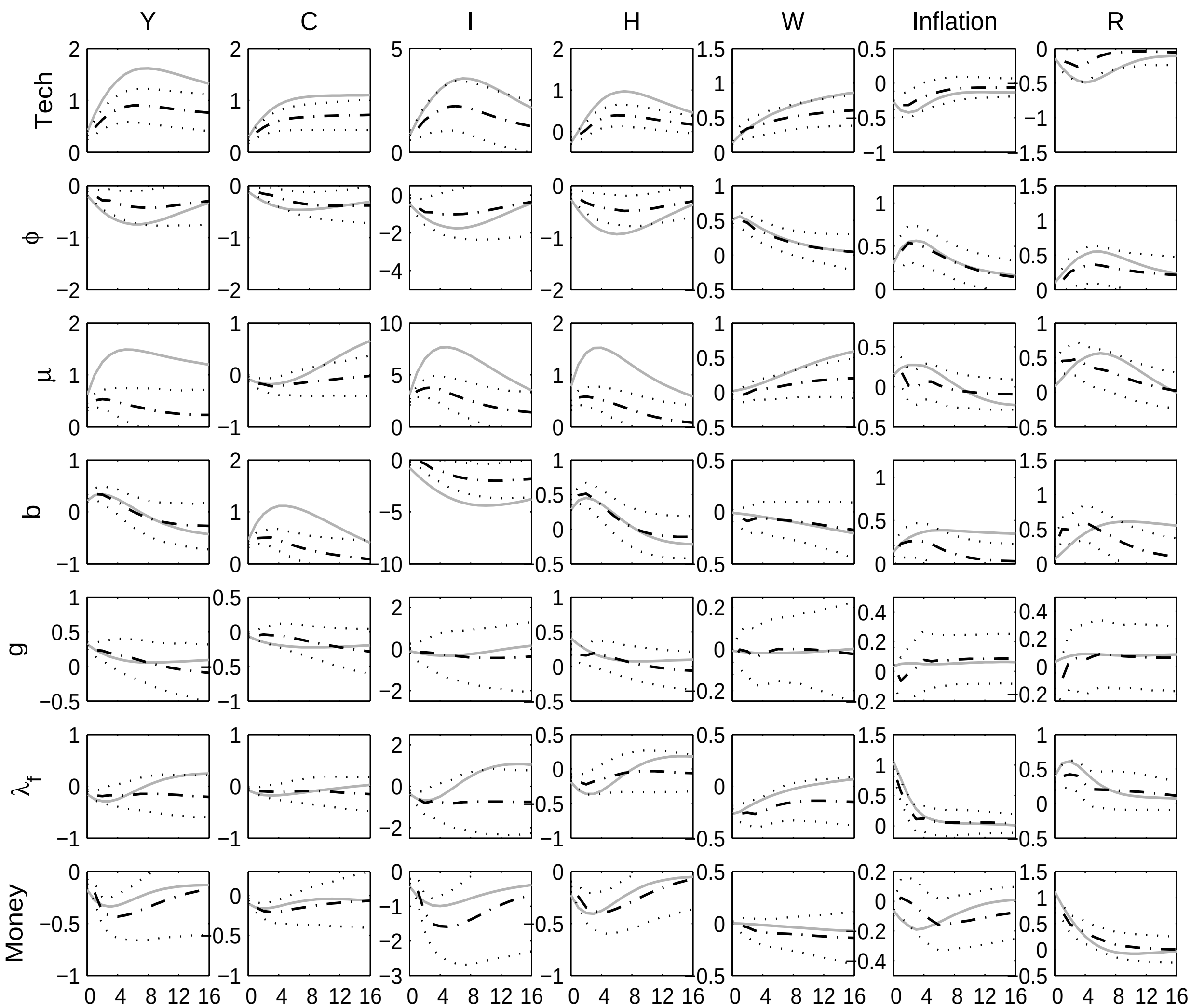
The impulse responses are computed with respect to the following shocks: technology growth z_t (*Tech*), labor/leisure preference (φ), capital adjustment (μ), intertemporal preference (b), government spending (g), mark-up (λ_f), and monetary policy (*Money*).

Figure A-1: BASELINE MODEL IMPULSE RESPONSE FUNCTIONS: DSGE MODEL VS. DSGE-VAR(∞)



Notes: Figure depicts the impulse responses of the endogenous variables to one-standard deviation shocks for the DSGE-VAR(∞) (dotted lines) and for the state-space representation of the DSGE model (solid lines).

Figure A-2: BASELINE MODEL IMPULSE RESPONSE FUNCTIONS: DSGE-VAR($\hat{\lambda}$) VS. DSGE-VAR(∞)



Notes: Figure depicts mean responses of the endogenous variables to one-standard deviation shocks for the DSGE-VAR(∞) (gray solid lines), the DSGE-VAR($\hat{\lambda}$) (dark dash-and-dotted lines), and 90% bands (dark dotted lines) for DSGE-VAR($\hat{\lambda}$).