

The Zero Lower Bound and a Neo-Classical Phillips Curve

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April 20, 2010

Abstract

With sticky prices, optimizing agents and money in the utility function, I derive the exact analytical solution for optimal monetary policy given a zero lower bound (ZLB) on the interest rate. The Phillips curve is Neo-Classical, and the ZLB is then not a constraint on optimal policy. Optimal policy is history dependent even without a commitment problem and implements a Friedman rule equilibrium. Policy rule parameters, like the response to inflation, are not identified under optimal policy. The optimal policy rule intercept term is time varying and depends on the variance of the natural real rate.

*For helpful comments to an earlier version of this paper, I thank Anton Braun, Matthew Canzoneri, Behzad Diba, Dale W. Henderson, Preston Miller, Øistein Røisland, Tommy Sveen and participants in seminars at the Federal Reserve Board and Norges Bank (the Central Bank of Norway) and at the CREI/Bank of Canada conference on October 10-11th 2008, "New Perspectives on Monetary Policy Design". Remaining errors are my own. The views in this paper are solely the responsibility of the author and do not necessarily reflect those of Norges Bank.

1 Introduction and summary

I use a model with sticky prices, optimizing agents and money in the utility function to derive the exact analytical solution for optimal monetary policy given a zero lower bound on the nominal interest rate¹. My model has a Neo-Classical Phillips curve, unlike the literature that has emerged on this topic since Krugman (1998) revived it². I find that the main result on optimal policy when there is a zero lower bound is robust to the specification of the Phillips curve: optimal monetary policy is history dependent. This applies even without any commitment problem.

Some agents set prices flexibly, while some agents set prices one period in advance. This creates a Neo-Classical Phillips curve and enables me to solve the model analytically without imposing certainty equivalence, following Henderson and Kim (2001). The model may be viewed as a microfounded and fully intertemporal version of Krugman (1998). The consumption Euler equation drives the result that future inflation helps stabilize the economy when the natural real rate is negative: The real rate can be lowered at the zero lower bound only through higher inflation in later periods. This is the same kind of history dependence in optimal policy as Eggertsson and Woodford (2003), Eggertsson and Woodford (2004) and Krugman (1998) describe. This history dependence differs from history dependence describing optimal time-inconsistent policy in e.g. Woodford (2003b), in that monetary policy in the zero lower bound-papers depends on a lagged exogenous variable - the natural interest rate.

Eggertsson and Woodford (2003) solve their model numerically. The policymaker in their model may postpone inflation and work through lower expected real interest rates further into the future in order to stimulate demand today, because of a New-Keynesian Phillips curve. But the trigger for this policy is a low natural real rate at some point. Since variation in expected inflation is costly with a New-Keynesian Phillips curve, there is a trade-off and optimal policy is time-inconsistent. The Neo-Classical Phillips curve in my model implies that the zero lower bound is not a binding constraint on optimal monetary policy: When known variation in the rate of inflation is not a concern, the zero lower bound also ceases to be a concern. It is first best optimal to implement the Friedman rule and leave the nominal interest rate

¹The lower bound on nominal interest rates need not be exactly zero. Costly storage of money would make a negative lower bound possible. Nonsatiation in real money balances would create a positive lower bound. Like much of the literature, this paper refers to a zero lower bound for simplicity.

²Adam and Billi (2006) and (2007), Billi (2005) and (2007), Wolman (1998), Wolman (2005), Fuhrer and Madigan (1997), and Rotemberg and Woodford (1997) are some of the important contributions to the literature on interest rate policy and the zero lower bound. A different strand of the zero lower bound literature concerns alternative instruments for use when there is a zero lower bound. Examples are Svensson (2001), Clouse, Henderson, Orphanides, Small, and Tinsley (2003), Orphanides and Wieland (2000) and Curdia and Woodford (2009a). Yet another strand of the literature discusses a multiplicity problem related to the nonlinearity introduced by the zero lower bound, e.g. Benhabib, Schmitt-Grohe, and Uribe (2001a) and Alstadheim and Henderson (2006).

marginally above zero in equilibrium - and this first best solution is implementable. I assume that the support of the shock to the natural real rate is bounded, so that there always exists an inflation rate high enough to equate the real interest rate to the natural real rate, even with a zero lower bound on the nominal interest rate.

While confirming the history-dependence-result, my model still illustrates how the role of forward guidance in monetary policy in a model with fully rational and forward-looking optimizing agents may be limited: Monetary policy has no leverage over the real interest rate beyond the next period, because of the Neo-Classical Phillips curve. Hence, even if agents are forward-looking, the policymaker does not have the option to postpone expansionary policy beyond the next period. On the other hand, the policymaker here is quite happy to inflate in the next period, because expected inflation is not costly.

The price level is a random walk and the inflation rate is stationary under optimal policy. While monetary policy establishes a nominal anchor in the sense that the initial price level is determined by including inflation in the monetary policy reaction function, it does not follow in my model that monetary authorities should aim to stabilize inflation in equilibrium. If authorities suboptimally should choose to stabilize the inflation rate, the zero lower bound would become a constraint, as in the model with a New-Keynesian Phillips Curve.

I show that the identification challenge highlighted in Cochrane (2007) also applies with a Neo-Classical Phillips curve. Under optimal policy, the parameter that ensures determinacy by responding strongly enough to inflation is not identified. A suboptimal reaction function is required in order to identify the parameter securing determinacy. Cochrane showed this in a flexible price setup as well as in a setup with a New-Keynesian Phillips curve. Also, the optimal policy rule responds directly to shocks, as in Woodford (2001), and cannot be implemented via policy response to endogenous variables alone. Hence, the intercept term in the optimal policy rule has to be time-varying. In my model it also depends on the distribution of the shock to the natural real rate, since I do not impose certainty equivalence.

The next section describes the model. In section 3, I solve the flexible-price version of the model and derive an optimal interest-rate rule in that case. Next, in section 4, I solve the sticky-price version of the model. Section 5 presents an interest-rate rule that implements the first best solution in the case with preset prices. Section 6 provides concluding remarks.

2 The model

The model follows Aoki (2001) in that it both has a log-linear aggregate price index and one sector with sticky prices and one sector with flexible prices. The economy is

closed. Agents in the sector with sticky prices set prices one period in advance. This price setting means that the model can be solved exactly using the method of undetermined coefficients, as in Henderson and Kim (2001). Agents are yeoman farmers, there is no use of capital in production, and no government consumption. Agents derive utility from real money balances. Money is superneutral in the flexible-price version of the model in the sense that inflation does not affect output. In the sticky-price version of the model, money is not super-neutral, in the sense that expected inflation in the next period affects output today. Money does not pay interest in this model, and hence a zero nominal interest rate would eliminate distortions to money holdings³.

A representative agent maximizes the following objective with respect to consumption c , her output price p and money m and bonds b , subject to a period budget constraint, where λ_t is the langrange multiplier on the constraint:

$$\begin{aligned} \text{Max } E_n \left\{ \sum_{t=n}^{\infty} \beta^{t-n} \left(\frac{c_t^{1-\rho} - 1}{1-\rho} \right) - \frac{1}{2} \kappa_t y_t^2 + f\left(\frac{m_t}{P_t}\right) + \lambda_t [(1+\omega)p_t y_t + m_{t-1} + \right. \\ \left. (1+i_{t-1})b_{t-1}^g + b_{t-1} - t_t - P_t c_t - m_t - b_t^g - \delta_{t,t+1} \mathbf{b}_t] \right\} \end{aligned} \quad (1)$$

where

$$c_t = \frac{c_{s,t}^\gamma c_{f,t}^{1-\gamma}}{\gamma^\gamma (1-\gamma)^{1-\gamma}}, \quad (2)$$

$$c_{s,t} \equiv \left[\int_{j=0}^1 (c_{js,t})^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}, \quad c_{f,t} \equiv \left[\int_{i=0}^1 (c_{if,t})^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \quad (3)$$

and

$$f\left(\frac{m_t}{P_t}\right) = \begin{cases} -\frac{1}{2} \mu (\delta - \frac{m_t}{P_t})^2, & \frac{m_t}{P_t} \leq \delta \\ 0, & \frac{m_t}{P_t} > \delta \end{cases} . \quad (4)$$

j indexes producers of different period t 'sticky-price' (s) goods $c_{js,t}$. $c_{s,t}$ is the composite sticky-price good. i indexes the 'flexible-price' (f) goods ($c_{if,t}$) while $c_{f,t}$ is the composite flexible-price good. c_t is the composite consumption good for period t that goes into each agent's period utility function. The term $-\frac{1}{2} \kappa_t y_t^2$ represents disutility from producing output in period t . κ is an i.i.d. negative supply shock

³The distortion following a high and variable nominal rate in my model may be viewed as a proxy for the cost of high average inflation and a preference for interest rate smoothing. In practise, many central banks pay interest on money. That is, the policy relevant part of base money, central bank reserves, pay interest. Interest payments on central bank reserves is modelled in e.g. Curdia and Woodford (2009b). When central banks pay interest on reserves, the Friedman rule does not apply. An adjusted version of the Friedman rule would apply to the difference between the short term market rate and the central bank deposit rate. This adjusted Friedman rule would say that if the central bank costlessly could provide more reserves (e.g. by holding claims that covered the costs of paying interest on reserves), the short term market rate should be brought down to the floor established by the central bank deposit rate. That would relieve the banks of the cost of managing their reserves - today's "shoe-leather costs". This is an argument for a "floor-system" rather than "corridor-system" for liquidity management.

common to both sectors⁴, with a mean equal to one. It will determine the natural real rate of the model. The last term in the period utility function represents utility from holding real money balances. P_t is the price of c_t in terms of m_t . The inverse of μ in equation (4) is proportional to the interest elasticity of money demand.

Each agent maximizes utility subject to the constraint that income from production after taxes or subsidies, $(1 + \omega)p_t y_t$, plus financial assets and their return brought over from last period (money m_{t-1} , bonds b_{t-1} and $(1 + i_{t-1})b_{t-1}^g$) must equal taxes t_t , consumption expenditure $P_t c_t$ and new holdings of financial assets. b_t^g is the nominal value of risk free government bonds, while \mathbf{b}_t is a vector of quantities of state contingent claims, and $\boldsymbol{\delta}_{t,t+1}$ is the vector of the prices of those claims. Each state contingent claim pays one unit of currency in the subsequent period given a particular realization of the state in that period. The gross risk free nominal interest rate, $1 + i_t$ (I will also use I_t for this variable) is therefore equal to $[\boldsymbol{\delta}_{t,t+1} \cdot \mathbf{1}]^{-1}$, where $\mathbf{1}$ is a vector of ones.

2.1 The intratemporal problem and goods market equilibrium

This subsection presents the goods market equilibrium. Some more details are provided in appendix A on page 19.

I ignore time subscripts here. I assume that there are complete markets and perfect risk sharing⁵. All consumers consume the same amount $c = C$ of the aggregate consumption index, and also the same amounts of the indexes of sticky-price consumption and flexible-price consumption, $c_s = C_s$ and $c_f = C_f$. The total demand from consumers i that faces producer j in the sticky-price sector is⁶

$$y_s = \int_{i=0}^2 (c_{js}) di = \int_{i=0}^2 \left[\left(\frac{P_{js}}{P_s} \right)^{-\theta} c_s \right] di = \left(\frac{P_{js}}{P_s} \right)^{-\theta} 2C_s = \left(\frac{P_{js}}{P_s} \right)^{-\theta} 2 \left(\frac{P_s}{P} \right)^{-1} \gamma C, \quad (5)$$

where $2C_s \equiv \int_{i=0}^2 c_s di$. I have used that $C_s = \left(\frac{P_s}{P} \right)^{-1} \gamma C$ where P_s is the sticky-price goods price index. The corresponding expression applies for the demand for flexible-price goods, but with $(1 - \gamma)$ instead of γ :

$$y_f = \int_{i=0}^2 (c_{jf}) di = \int_{i=0}^2 \left[\left(\frac{P_{jf}}{P_f} \right)^{-\theta} c_f \right] di = \left(\frac{P_{jf}}{P_f} \right)^{-\theta} 2C_f = \left(\frac{P_{jf}}{P_f} \right)^{-\theta} 2 \left(\frac{P_f}{P} \right)^{-1} (1 - \gamma) C. \quad (6)$$

⁴In a yeoman farmer model, the labor market is internalized. κ may be interpreted as a labor supply shock or a productivity shock. In particular, following Obstfeld and Rogoff (1996), the productivity variable κ may be understood as follows: Let disutility from work effort l be given by $-\phi l$ and the production function be Al^α , $\alpha < 1$. Inverting the production function gives $l = \left(\frac{y}{A} \right)^{1/\alpha}$. Given $\alpha = \frac{1}{2}$ and $\kappa = \frac{2\phi}{A^{1/\alpha}}$, we get $-\phi \left(\frac{y}{A} \right)^{1/\alpha} = -\frac{1}{2} \kappa y^2$.

⁵Agents learn which sector they will belong to in the next period at the point in time when the sticky-price agents need to set their preset price. They buy state contingent claims before they learn which sector they belong to.

⁶Note that the total mass of agents is two, there is a mass one of sticky price producers and a mass one of flexible price producers, and all are also consumers.

Symmetry across producers within the same sector implies that they all will set the same price and produce the same amount. (5) and (6) together with goods market equilibrium imply that (using capital Y for aggregate output)

$$Y_s = 2C_s = \left(\frac{P_s}{P}\right)^{-1}\gamma 2C \quad (7)$$

and

$$Y_f = 2C_f = \left(\frac{P_f}{P}\right)^{-1}(1 - \gamma)2C. \quad (8)$$

From $Y \equiv \frac{(Y_s)^\gamma(Y_f)^{1-\gamma}}{\gamma^\gamma(1-\gamma)^{1-\gamma}}$ and $P = P_s^\gamma P_f^{1-\gamma}$,

$$Y = 2C. \quad (9)$$

2.2 The intertemporal problem

I use the expression for demand for each individual producers good (5) and (6) and the fact that all consumers will consume the same (since there is perfect risk sharing). I let small letters denote individual variables. I ignore the i, j indexing of individuals since agents in the same sector set the same price. Substituting into the objective function (1) gives the following version of the problem of a representative agent in the sticky-price (s) sector:

$$\begin{aligned} \text{Max } E_n \left\{ \sum_{t=n}^{\infty} \beta^{t-n} \left\{ \left(\frac{c_t^{1-\rho} - 1}{1-\rho} \right) - \frac{1}{2} \kappa_t \left[\left(\frac{p_{s,t}}{P_{s,t}} \right)^{-\theta} 2 \left(\frac{P_{s,t}}{P_t} \right)^{-1} \gamma C_t \right]^2 + f\left(\frac{m_t}{P_t}\right) \right. \right. \\ \left. \left. + \lambda_t \left[(1 + \omega) \left[\left(\frac{p_{s,t}}{P_{s,t}} \right)^{-\theta} 2 \left(\frac{P_{s,t}}{P_t} \right)^{-1} \gamma C_t \right] + m_{t-1} + \right. \right. \right. \\ \left. \left. \left. (1 + i)_{t-1} b_{t-1}^g + b_{t-1} - t_t - P_t c_t - m_t - b_t^g - \delta_{t,t+1} \mathbf{b}_t \right] \right\} \right\}. \end{aligned} \quad (10)$$

Agents maximize the objective with respect to the consumption index c_t , bonds b_t^g and \mathbf{b}_t , money m_t and the price $p_{s,t}$ of their output. $p_{s,t}$ denotes a price set in period $t - 1$ (s for sticky) that applies in period t . The objective of an agent in the flexible-price sector is equal to the one above, except that $p_{f,t}$ replaces $p_{s,t}$. $p_{f,t}$ is a price set in period t that also applies in period t . The following additional constraints must apply in order for the problem to be well defined: $c_t > 0$, $m_t > 0$, $p_t > 0$, $\forall t$, and $\theta > 1$. $\text{Pr}(\varepsilon)$ denotes the probability of state ε . Differentiating with respect to the composite consumption good and assets gives

$$\lambda_t = \frac{1}{P_t c_t^\rho}, \quad (\text{consumption}) \quad (11)$$

$$\lambda_t = E_t \{ \beta(1 + i_t)(\lambda_{t+1}) \}, \quad (\text{risk free bonds}) \quad (12)$$

$$\beta \lambda_{t+1} = \lambda_t \frac{\delta_{t,t+1}(\varepsilon)}{\text{Pr}(\varepsilon)}, \quad (\text{state-dependent bonds}) \quad (13)$$

and

$$f'(\frac{m_t}{P_t}) = E_t \{ \lambda_t P_t - \beta \lambda_{t+1} P_t \}. \quad (\text{money}) \quad (14)$$

Differentiating with respect to the price that will apply one period ahead gives

$$\begin{aligned} & E_t \left\{ \kappa_{t+1} \theta \left(\frac{P_{s,t+1}}{P_{s,t+1}} \right)^{-2\theta} \left(\frac{P_{s,t+1}}{P_{t+1}} \right)^{-2} C_{t+1}^2 2^2 \gamma^2 \frac{1}{p_{s,t+1}} \right\} \\ &= -E_t \left\{ \lambda_{t+1} (1+s) \left(\frac{P_{s,t+1}}{P_{t+1}} \right)^{-1} 2\gamma C_{t+1} [1-\theta] \left(\frac{P_{s,t+1}}{P_{s,t+1}} \right)^{-\theta} \right\}. \end{aligned} \quad (15)$$

Simplifying (15), lagging one period and substituting for the expression for $y_{s,t}$ in terms of relative prices gives

$$p_{s,t} = \frac{\frac{\theta}{\theta-1}}{(1+\omega)} \frac{E_{t-1} \{ \kappa_t y_t^{s2} \}}{E_{t-1} \{ y_t^s \lambda_t \}}. \quad (\text{sticky price}) \quad (16)$$

This is the marginal rate of substitution between the disutility from production and the utility from consumption, adjusted for market power (decreasing as θ increases) and subsidies s . In the flexible-price case I get the following equation instead of (16):

$$p_{f,t} = \frac{\frac{\theta}{\theta-1}}{1+\omega} \frac{\kappa_t y_f}{\lambda_t}. \quad (\text{flexible price}) \quad (17)$$

Using (11) and (12) I get

$$c_t^{-\rho} = E_t \left\{ \frac{\beta(1+i_t)}{P_{t+1}/P_t} c_{t+1}^{-\rho} \right\}. \quad (\text{Euler equation}) \quad (18)$$

Agents are also subject to a no-Ponzi-game condition,

$$E_t \left\{ \lim_{s \rightarrow \infty} \frac{m_{t+s} + b_{t+s}}{P_{t+s}} \prod_{j=t}^{j=s} (1+i_j)^{-1} \geq 0 \right\}. \quad (\text{No-Ponzi-game condition}) \quad (19)$$

The model can be solved for output and inflation independently of real money balances. But money is relevant for welfare evaluation. For future reference (11), (14) and (18) give money demand characterized by⁷

$$\frac{1}{\mu} \frac{i_t}{1+i_t} (c_t^{-\rho}) = \begin{cases} \delta - \frac{m_t}{P_t} & \text{for } \frac{m}{P} \leq \delta \\ 0 & \text{for } \frac{m}{P} > \delta \end{cases}. \quad (20)$$

Condition (20) translates into a constraint on the equilibrium nominal interest rate⁸:

$$\frac{i_t}{1+i_t} \geq 0, \quad (21)$$

⁷The interest elasticity of money demand is $\frac{d \frac{m}{P}}{d \frac{i}{1+i}} \frac{\frac{i}{1+i}}{\frac{m}{P}} = -\frac{1}{\mu} C^{-\rho} \frac{\frac{i}{1+i}}{\frac{m}{P}}$

⁸If $f'(\cdot)$ had been specified to be negative for large enough $\frac{M}{P}$, the zero lower bound would apply anyway in equilibrium. In that case, $f'() = 0$ would have described the unique first best quantity of money. In the present case, the first best $\frac{M}{P}$ is not unique, but instead described by the open interval from δ and up. One could have specified $f'()$ to reach some lower bound above zero. In that case the lower bound on the interest rate would have been strictly positive. One way to get a negative equilibrium nominal interest rate would be to introduce storage cost on money (e.g. tax on money holdings). This would give money a character of being 'perishable', and perishable goods can have negative nominal interest rates.

since $c > 0$ and $\mu > 0^9$.

2.3 A summary of the equilibrium conditions

Let $1+\omega = \frac{\theta}{\theta-1}$, so that the effect of monopolistic competition on output is eliminated. Using goods market equilibrium,

$$Y_f = 2C_f, \quad Y_s = 2C_s \quad \text{and} \quad Y = \frac{(Y_s)^\gamma (Y_f)^{1-\gamma}}{\gamma^\gamma (1-\gamma)^{1-\gamma}} = 2C,$$

and symmetry across producers so that $p_{s,t} = P_{s,t}$ and $p_{f,t} = P_{f,t}$, I have the following unknown variables: $P_{s,t}, P_{f,t}, P_t, Y_t, Y_{s,t}, Y_{f,t}, i_t, \lambda_t$, while real money balances can be determined recursively. The first order conditions in terms of aggregate variables are

$$P_{s,t} = \frac{E_{t-1}\{\kappa_t Y_{s,t}^2\}}{E_{t-1}\{Y_{s,t} \lambda_t\}}, \quad (\text{sticky price}) \quad (22)$$

$$P_{f,t} = \frac{\kappa_t Y_{f,t}}{\lambda_t}, \quad (\text{flexible price}) \quad (23)$$

$$\lambda_t = \frac{(\frac{1}{2}Y_t)^{-\rho}}{P_t}, \quad (\text{consumption}) \quad (24)$$

$$P_t = P_{s,t}^\gamma P_{f,t}^{1-\gamma}, \quad (\text{price equation}) \quad (25)$$

$$\left(\frac{1}{2}Y_t\right)^{-\rho} = E_t\left\{\frac{\beta(1+i_t)}{P_{t+1}/P_t} \left(\frac{1}{2}Y_{t+1}\right)^{-\rho}\right\}, \quad (\text{demand}) \quad (26)$$

$$Y_{f,t} = \left(\frac{P_{f,t}}{P_t}\right)^{-1} (1-\gamma) Y_t, \quad (\text{flexible-price output}) \quad (27)$$

and

$$Y_{s,t} = \left(\frac{P_{s,t}}{P_t}\right)^{-1} \gamma Y_t. \quad (\text{sticky-price output}) \quad (28)$$

I eliminate time subscripts (using +1 and -1 for leads and lags), eliminate λ , define $\frac{P}{P_{-1}} = \Pi$ and divide the price equations by P_{-1} . This gives me a system in terms of the inflation rate, including the following two equations:

$$\frac{P_s}{P_{-1}} = \frac{E_{-1}\{\kappa Y_s^2\}}{E_{-1}\{Y_s (\frac{1}{2}Y)^{-\rho} \Pi^{-1}\}} \quad (\text{sticky price}) \quad (29)$$

and

$$\frac{P_f}{P_{-1}} = \frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}} \Pi. \quad (\text{flexible price}) \quad (30)$$

⁹Condition (21) could technically be satisfied also if $i < -1$, but that would violate the Euler equation.

I substitute out for relative prices (29) and (30) in equations (25), (27) and (28), and arrive at the following system of four equations:

$$\Pi = \left[\frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}} \Pi \right]^{1-\gamma} \left[\frac{E_{-1}(\kappa Y_s^2)}{E_{-1}(Y_s(\frac{1}{2}Y)^{-\rho} \Pi^{-1})} \right]^\gamma, \quad (\text{price equation}) \quad (31)$$

$$\left(\frac{1}{2}Y\right)^{-\rho} = \beta(1+i)E(\Pi_{+1}^{-1}\{\frac{1}{2}Y_{+1}\}^{-\rho}), \quad (\text{demand}) \quad (32)$$

$$Y_f = \left(\frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}}\right)^{-1}(1-\gamma)Y, \quad (\text{flex-price output}) \quad (33)$$

and

$$Y_s = \left[\left(\frac{E_{-1}\{\kappa Y_s^2\}}{E_{-1}\{Y_s(\frac{1}{2}Y)^{-\rho} \Pi^{-1}\}} \right)^{-1} \Pi \gamma Y \right]. \quad (\text{sticky-price output}) \quad (34)$$

Equations (31)-(34) may be used together with some specification for monetary policy to solve for Π , Y_s , Y_f , Y and $1+i$.

2.4 Monetary and fiscal policy

I assume that authorities use the following interest rate rule:

$$1+i = I^* \beta^{-1} \kappa^{\lambda_\kappa} \kappa_{-1}^{\lambda_{\kappa-1}} \Pi^{\lambda_\pi}. \quad (\text{interest-rate rule}) \quad (35)$$

It is convenient to let authorities use a log-linear interest-rate rule. The model may then be solved analytically with linear tools, while the zero lower bound constraint is not violated as long as the support of the shock is bounded. While only considering linear policy may seem like a restriction, it turns out not to be - the first best allocation is attainable with this rule; both in the flexible price case and in the sticky price case. My approach is to derive the first best allocation and then back out the policy parameters that support that allocation.

The rule is Taylor-type, but without response to the output gap, and with a time-variable intercept term, given by $I^* \beta^{-1} \kappa^{\lambda_\kappa} \kappa_{-1}^{\lambda_{\kappa-1}}$. $\lambda_\pi > 1$ ensures determinacy. Intuitively, response to κ may stand in for a response to the output gap, keeping in mind that κ captures the marginal cost of production in this model. Alternatively, note that the natural real interest rate - or the Wicksellian rate of interest - is variable according to κ . Woodford (2001) describes how a simple interest rate rule may implement optimal monetary policy as long as it includes a time-varying intercept term equal to the natural real rate.

In addition to the general rule (35), I consider a simple inflation-targeting rule, with λ_κ and $\lambda_{\kappa-1}$ both equal zero but λ_π is nonzero.

Utility maximization combined with the No-Ponzi-game condition (19), implies the transversality condition, which is that (19) holds with equality in equilibrium. I

assume throughout that (19) holds for all on or off equilibrium paths of endogenous variables - that is, fiscal policy is Ricardian. When initial net public debt $M_n + B_n$ is positive it will be satisfied with e.g. a balanced budget rule for fiscal policy and a nominal interest rate that is at least marginally positive with some positive probability in some periods. In order to rule out explosive or implusive price level paths respectively, one may want to add a condition that the government guarantees a minimal real redemption value of money (as in Obstfeld and Rogoff (1983)) and a condition that consolidated nominal government debt grows at a minimal rate should the price level embark on an implusive path¹⁰.

3 The flexible-price model

I first solve the model given flexible prices in both sectors. Since I do not impose certainty equivalence, the distribution of the shock matters. I want to work with a bounded support and I choose a uniform distribution for simplicity. Flexible prices in both sectors mean that the relative price is determined by the fixed parameter γ ¹¹;

$$\frac{P_s}{P_f} = \left\{ \frac{\gamma}{1-\gamma} \right\}^{\frac{1}{2}}, \quad (36)$$

and output in the two sectors are given by

$$Y_f = \kappa^{-1} \left\{ \frac{\gamma}{1-\gamma} \right\}^{-\frac{1}{2}\gamma} \left\{ \frac{1}{2}Y \right\}^{-\rho}, \quad Y_s = \kappa^{-1} \left\{ \frac{\gamma}{1-\gamma} \right\}^{\frac{1}{2}(1-\gamma)} \left\{ \frac{1}{2}Y \right\}^{-\rho}. \quad (37)$$

Substituting the above into

$$Y \equiv \frac{(Y_s)^\gamma (Y_f)^{1-\gamma}}{\gamma^\gamma (1-\gamma)^{1-\gamma}}$$

implies that

$$Y = K \cdot \kappa^{-\frac{1}{1+\rho}}, \quad (38)$$

$K = \left(\frac{1}{(1-\gamma)^{1-\gamma} \gamma^\gamma} \right)^{\frac{1}{1+\rho}} \left(\frac{1}{2} \right)^{-\frac{\rho}{1+\rho}}$, and

$$C = \frac{1}{2}Y = \frac{1}{2}K \cdot \kappa^{-\frac{1}{1+\rho}}. \quad (39)$$

In the symmetric case¹² where $\gamma = \frac{1}{2}$ I will have $\frac{P_s}{P_f} = 1$, $C = \kappa^{-\frac{1}{1+\rho}}$ and $Y = 2\kappa^{-\frac{1}{1+\rho}}$.

¹⁰See e.g. section 4.2 in chapter 2 of Woodford (2003a)

¹¹Use equations (22), (23), (27) and (28) to see this.

¹²Given a fixed and equal mass of agents in the sticky-price sector and the flexible-price sector, it seems most natural to consider the symmetric case. If demand was skewed in the direction of one sector, one would think that more producers would move to that sector. I will keep the γ parameter general in the solutions I derive, though. There might be other applications of the model where a variable γ could be interesting to consider. E.g. one might apply the model to a monetary union, where one country had sticky prices and one country had flexible prices.

As $\rho \rightarrow 0$, the agents are risk neutral (the coefficient of relative risk aversion is zero) and smoothing of consumption across different κ -values is less important. Hence output and consumption vary inversely with the supply shock. As $\rho \rightarrow \infty$, output and consumption are less responsive to κ , since consumption smoothing is more important. The special case of $\rho = 1$ is the case of log utility of consumption¹³.

With output given by equation (39), the unknown variables are the inflation rate and the nominal interest rate. The demand equation and the interest-rate rule now give me a system of two equations in two unknown, Π and I ,

$$\kappa^{\frac{\rho}{1+\rho}} = \beta(1+i)E(\Pi_{+1}^{-1}\kappa_{+1}^{\frac{\rho}{1+\rho}}) \quad (\text{demand}) \quad (40)$$

and

$$1+i = I^*\beta^{-1}\kappa^{\lambda\kappa}\kappa_{-1}^{\lambda\kappa-1}\Pi^{\lambda\pi}. \quad (\text{interest-rate rule}) \quad (41)$$

$\lambda_\pi > 1$ is necessary in order to ensure a determinate initial inflation rate and hence a determinate initial price level¹⁴¹⁵. This can be seen by substituting the expression for the interest rate from the interest rate rule into the demand equation. The resulting difference equation in inflation has a unique solution for the path of the inflation rate only if $|\lambda_\pi| > 1$, and I will assume λ_π to be positive. Note that this result holds in the exact version of the model - it is a global result. The indeterminacy question is the same in the flexible price version of the model as in the sticky price model below, and it is independent of the specification of the Phillips curve¹⁶. The source of the indeterminacy problem is a demand equation like equation 40.

3.1 The flexible-price model solution.

In order to solve (40) and (41) I follow Henderson and Kim (2001) and use the method of undetermined coefficients. The guess for the solution for the inflation rate is

$$\Pi = \Phi\kappa^{\phi\kappa}\kappa_{-1}^{\phi\kappa-1}. \quad (\text{guess for } \Pi) \quad (42)$$

In appendix B on page 20, I show that (42) is a solution of (40) and (41) with parameters as given in table 1. κ has a uniform distribution between κ_L and κ_H .

¹³The coefficient ρ also represents the inverse of the intertemporal elasticity of substitution.

¹⁴Equations 40 and 41 determine a unique (expected) inflation rate as of period 1, P_1/P_0 , given any initial P_0/P_{-1} and any λ_π . The inflation rate in period 0, P_0/P_{-1} , which given P_{-1} pins down today's price level P_0 and the rest of the price path, is not given by the two equations, however. I use the standard approach, and assume that we may rule out any unstable inflation path. All but one P_0 will make the path of inflation explode or implode if we let $|\lambda_\pi| > 1$. Hence, I assume $\lambda_\pi > 1$.

¹⁵I ignore the possibility of a second steady state equilibrium where the inflation rate is lower than the solution that I find here, as considered in e.g. Benhabib, Schmitt-Grohe, and Uribe (2001b) and in Alstadheim and Henderson (2006). The potential multiplicity can be ruled out by an appropriate assumption about fiscal policy or by making the interest rate rule nonmonotonic in the inflation rate.

¹⁶The indeterminacy issue arises analogously when monetary policy is specified as a money supply rule. See section 4, Chapter 2 in Woodford (2003a), Alstadheim and Henderson (2006) and Obstfeld and Rogoff (1983).

Table 1: The flexible-price model solution	
$\phi_\kappa = \left[\frac{\rho}{1+\rho} - \lambda_\kappa + \phi_{\kappa-1} \right] \frac{1}{\lambda_\pi}$	(1.1)
$\phi_{\kappa-1} = -\frac{\lambda_{\kappa-1}}{\lambda_\pi}$	(1.2)
$\Phi = \beta \kappa_L^{-\lambda_\kappa - \phi_\kappa \lambda_\pi} \frac{1}{1 - \phi_\kappa + \frac{\rho}{1+\rho}} \left(\kappa_H^{1 - \phi_\kappa + \frac{\rho}{1+\rho}} - \kappa_L^{1 - \phi_\kappa + \frac{\rho}{1+\rho}} \right) \frac{1}{\kappa_H - \kappa_L}$	(1.3)
Memo: Equilibrium nominal interest rate: $1 + i = \left(\frac{\kappa}{\kappa_L} \right)^{(\lambda_\kappa + \phi_\kappa \lambda_\pi)} \geq 1$	

The authorities will prefer to let the policy variable I^* be as close to one as possible given any solution for Φ , $\phi_{\kappa-1}$ and ϕ_κ , in order to minimize money-demand distortions. Hence, I let I^* be tied down by the zero lower bound constraint - it should be exactly binding at the minimum value for κ .

It turns out that

$$I^* \equiv \beta \kappa_L^{-\lambda_\kappa - \phi_\kappa \lambda_\pi} \Phi^{-\lambda_\pi} \quad (43)$$

minimizes the nominal interest rate given the solution of the model (see appendix B). As indicated in table 1, the equilibrium nominal interest rate reaches its minimum when $\kappa = \kappa_L$, and its maximum when $\kappa = \kappa_H$. This is true as long as $\lambda_\kappa + \phi_\kappa \lambda_\pi \geq 0$, which it will be in all the cases that I consider. Intuitively, the nominal interest rate might need to be relatively high when the shock takes on a high value, because then productivity is expected to increase (κ is expected to fall), consumption and potential output are expected to increase, and this situation is characterized by a relatively high natural real interest rate.

If the interest rate responds to inflation only (and λ_π is bounded) the contemporaneous inflation rate will be

$$\Pi = \beta \left(\frac{\kappa}{\kappa_L} \right)^{\frac{\rho}{1+\rho}} \quad (44)$$

and the nominal interest rate

$$1 + i = \left(\frac{\kappa}{\kappa_L} \right)^{\frac{\rho}{1+\rho}}. \quad (45)$$

Hence, the nominal interest rate fluctuates in accordance with the natural real rate in equilibrium. In order for the demand equation (40) to hold, it has to, as long as *expected* inflation does not vary to produce the natural real interest rate. But this means that *current* inflation also has to vary in order to be consistent with the interest rate rule.

The equilibrium solution just described in (44) and (51) does not depend on λ_π , the parameter that governs determinacy. Hence, the example illustrates the point stressed by Cochrane (2007): only out-of-equilibrium observations could help identify the policy parameter governing determinacy in a simple case like this.

This identification issue does not depend on the Phillips curve. It follows from the forward-looking demand equation, as the one here based on the consumption Euler equation, and it appears both in the flexible price case and sticky price case studied below.

3.2 First-best policy in the flexible-price model

Since output is unaffected by monetary policy in the flexible-price model, authorities only need to minimize money-demand distortions in order to maximize welfare. Recall that the distortion from monopolistic competition is corrected by production subsidies. From equation (20) I know that the efficient level of real money balances, $\frac{M}{P} \geq \delta$, is reached when $i = 0$. From table 1 I know that I need

$$1 + i = \left(\frac{\kappa}{\kappa_L}\right)^{(\lambda_\kappa + \phi_\kappa \lambda_\pi)} = 1. \quad (46)$$

For this to hold for all κ , I need

$$\lambda_\kappa = -\phi_\kappa \lambda_\pi. \quad (47)$$

Given equation 1.1 in table 1, this implies

$$\lambda_{\kappa-1} = \frac{\rho}{1 + \rho} \lambda_\pi, \quad (48)$$

while λ_κ and λ_π can be chosen according to (47). With this policy the equilibrium inflation rate is

$$\begin{aligned} \Pi &= \beta \frac{1}{1 + \frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}} \left(\kappa_H^{1 + \frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}} - \kappa_L^{1 + \frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}} \right) \frac{1}{\kappa_H - \kappa_L} \kappa^{-\frac{\lambda_\kappa}{\lambda_\pi}} \kappa_{-1}^{-\frac{\rho}{1+\rho}} \\ &= \beta E_{-1} \left(\kappa^{1 + \frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}} \right) \kappa^{-\frac{\lambda_\kappa}{\lambda_\pi}} \kappa_{-1}^{-\frac{\rho}{1+\rho}}. \end{aligned} \quad (49)$$

Using the law of iterated expectations, expected Π , as of period -1 , is given by

$$E_{-1}(\Pi) = \beta E_{-1} \left(\kappa^{\frac{\rho}{1+\rho}} \right) \kappa_{-1}^{-\frac{\rho}{1+\rho}} = \beta E_{-1} \left\{ \frac{\kappa^{\frac{\rho}{1+\rho}}}{\kappa_L} \right\} \left(\frac{\kappa_{-1}}{\kappa_L} \right)^{-\frac{\rho}{1+\rho}} \quad (50)$$

While the nominal interest rate is constant,

$$1 + i = 1. \quad (51)$$

Intuitively, authorities are able to stabilize the nominal interest rate by letting the expected inflation rate instead of the nominal interest rate move along with the lagged productivity shock; compare (44) and (44) to the two equations above. In order to achieve a constant nominal interest rate in equilibrium, the policymaker has to respond to the lagged productivity shock by letting $\lambda_{\kappa-1} = \frac{\rho}{1+\rho} \lambda_\pi$. The inflation rate may also vary with the contemporaneous shock, according to λ_κ , but the nominal interest rate is stable at zero regardless of this¹⁷. Intuitively, the nominal interest

¹⁷ I^* is in the case of $\lambda_\kappa > 0$ given by:

$$I^* \equiv \beta \left[\beta \frac{1}{1 + \frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}} \left(\kappa_H^{1 + \frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}} - \kappa_L^{1 + \frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}} \right) \frac{1}{\kappa_H - \kappa_L} \right]^{-\lambda_\pi} = \beta^{1-\lambda_\pi} [E(\kappa^{1 + \frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}})]^{-\lambda_\pi}$$

rate does not have to vary with the contemporaneous inflation rate. The reason is that inflation between yesterday and today is irrelevant in order for the natural real rate between today and tomorrow to be attained and the consumption Euler equation to hold.

In order to uniquely pin down real and nominal money balances authorities need to let the nominal interest rate be marginally positive instead of letting the nominal rate be exactly zero at any point. This can be achieved by letting I^* in equation (43) be

$$I^* = \beta \kappa_L^{-\lambda_\kappa - \phi_\kappa \lambda_\pi} \Phi^{-\lambda_\pi} (1 + \varepsilon), \quad (52)$$

where ε is an arbitrarily small but strictly positive number. With this I^* , the inflation rate is

$$\Pi = \beta(1 + \varepsilon) E(\kappa^{-\phi_\kappa + \frac{\rho}{1+\rho}}) \kappa^{\phi_\kappa} \kappa_{-1}^{-\frac{\rho}{1+\rho}} \quad (53)$$

in equilibrium. Hence, the gross inflation rate is on average slightly higher than β when the nominal rate is set marginally above zero.

Note also that the no-Ponzi-game condition (19) rules out an equilibrium where the nominal interest rate stays exactly at zero at all times unless consolidated nominal public debt - fiscal and monetary authorities' debt taken together, or $m + b$ - is decreasing. With constant or growing nominal public debt there has to be some probability of positive nominal rates in some periods in equilibrium, see for example Benhabib, Schmitt-Grohe, and Uribe (2001a) and Alstadheim and Henderson (2006).

The λ_π parameter does not appear in the first best solution for inflation, output and the interest rate. Only with suboptimal choices for λ_κ and/or $\lambda_{\kappa-1}$, it will appear. Hence, λ_π cannot be identified under optimal policy in the flexible price solution..

3.3 Strict inflation targeting in the flexible-price model

Inflation targeting has practical interest and strict inflation targeting is one way of achieving the first-best level of output in the sticky-price version of the model. Here I will show how a stable inflation rate can be implemented in the flexible-price model. From the general solution for inflation in equation (42) together with table 1, I know that in order to stabilize the inflation rate perfectly I need $\phi_\kappa = \phi_{\kappa-1} = 0$. If authorities respond to shocks directly, they can let

$$\lambda_{\kappa-1} = 0 \text{ and } \lambda_\kappa = \frac{\rho}{1 + \rho}, \quad (54)$$

Inserting the above expression and the solution for $1 + i$ in (41) and using $\lambda_{\kappa-1} = \frac{\rho}{1+\rho} \lambda_\pi$ we get

$$1 + i = \beta^{1-\lambda_\pi} \left[E(\kappa^{1+\frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}}) \right]^{-\lambda_\pi} \cdot \beta^{-1} \kappa^{\lambda_\kappa} \kappa_{-1}^{\frac{\rho}{1+\rho} \lambda_\pi} \left[\beta E_{-1}(\kappa^{1+\frac{\lambda_\kappa}{\lambda_\pi} + \frac{\rho}{1+\rho}}) \kappa^{-\frac{\lambda_\kappa}{\lambda_\pi}} \kappa_{-1}^{-\frac{\rho}{1+\rho}} \right]^{\lambda_\pi} = 1$$

in which case

$$\Pi = \Phi = \beta(1 + \varepsilon)\kappa_L^{-\frac{\rho}{1+\rho}} E(\kappa^{\frac{\rho}{1+\rho}}). \quad (55)$$

If authorities choose to let $\lambda_\pi \rightarrow \infty$, they can stabilize the inflation rate completely regardless of λ_κ and $\lambda_{\kappa-1}$ since $\phi_\kappa = \phi_{\kappa-1} = 0$ also in that case. The constant inflation rate would still be at the same level. As the variance of κ increases, or κ_L declines towards zero and we consider a mean preserving spread, the inflation rate with strict inflation targeting will explode, given $\rho > 0$. Strict inflation targeting means that the nominal interest rate is given by $1 + i = (\frac{\kappa}{\kappa_L})^{\frac{\rho}{1+\rho}}$, while the rate of inflation is constant and output is at its first best level. Again, the equilibrium solution does not identify λ_π . I know that a nominal rate equal to zero minimizes the distortion of money demand. Hence, strict inflation targeting is associated with a welfare loss unless $\rho = 0$ or κ is constant so that the natural real rate is constant.

4 The sticky-price model

I use the same approach as in the flexible-price case, but with price stickiness the model is no longer superneutral, in the sense that unexpected inflation affects output. I need to solve for inflation and output simultaneously. Recall the equilibrium conditions:

$$\Pi = \left[\frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}} \Pi \right]^{1-\gamma} \left[\frac{E_{-1}(\kappa Y_s^2)}{E_{-1}(Y_s (\frac{1}{2}Y)^{-\rho} \Pi^{-1})} \right]^\gamma, \quad (\text{price equation}) \quad (56)$$

$$Y^{-\rho} = \beta(1 + i)E(\Pi_{+1}^{-1} Y_{+1}^{-\rho}), \quad (\text{demand}) \quad (57)$$

$$Y_f = \left(\frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}} \right)^{-1} (1 - \gamma)Y, \quad (\text{supply in flex-price sector}) \quad (58)$$

and

$$Y_s = \left[\left(\frac{E_{-1}\{\kappa Y_s^2\}}{E_{-1}\{Y_s (\frac{1}{2}Y)^{-\rho} \Pi^{-1}\}} \right)^{-1} \Pi \gamma Y \right]. \quad (\text{supply in sticky-price sector}) \quad (59)$$

Below I simplify the model in order to get a pair of equations in Π and Y only. Next, I use the interest-rate rule and guesses for output and inflation solutions to solve using the method of undetermined coefficients.

4.1 Simplifying the price equation

In order to get the price equation in terms of aggregate output and inflation only, I derive the sticky-price sector output and the flexible-price sector output as functions of total output. Use (56) to substitute out $\frac{E_{-1}(\kappa Y_s^2)}{E_{-1}(Y_s(\frac{1}{2}Y)^{-\rho\Pi^{-1}})}$ in (59) to get

$$Y_s = [\Pi^{\frac{1}{\gamma}} \left[\frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}} \Pi \right]^{\frac{\gamma-1}{\gamma}}]^{-1} \Pi \gamma Y. \quad (60)$$

Rearrange (58) to get an expression for Y_f in terms of Y ,

$$Y_f = \kappa^{-\frac{1}{2}} \left(\frac{1}{2}\right)^{-\frac{\rho}{2}} Y^{\frac{1-\rho}{2}} (1-\gamma)^{\frac{1}{2}}. \quad (61)$$

Substituting out for Y_f in (60) and simplifying gives Y_s as a function of Y only,

$$Y_s = \kappa^{\frac{1-\gamma}{2\gamma}} \left(\frac{1}{2}\right)^{\frac{\rho(1-\gamma)}{2\gamma}} Y^{\left[\frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma}\right]} (1-\gamma)^{\frac{1-\gamma}{2\gamma}} \gamma. \quad (62)$$

Substituting for (61) and (62) in equation (56) gives

$$\Pi = \left[\kappa^{\frac{1}{2}} Y^{\frac{1+\rho}{2}} \right]^{\frac{1-\gamma}{\gamma}} (1-\gamma)^{\frac{1-\gamma}{\gamma}} \gamma \left(\frac{1}{2}\right)^{\frac{\rho(1-\gamma)}{\gamma} + \rho} \frac{E_{-1}(\kappa^{\frac{1}{\gamma}} Y^{\left[\frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma}\right]})}{E_{-1}(\kappa^{\frac{1-\gamma}{2\gamma}} Y^{\left[\frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma}\right]-\rho} \Pi^{-1})}. \quad (63)$$

This equation says that the inflation rate is determined by the expected inflation rate, actual output and expected output.

4.2 Solving the sticky-price model

I now have the price equation and the demand equation, and I add an interest-rate rule:

$$\Pi = (1-\gamma)^{\frac{1-\gamma}{\gamma}} \gamma \left(\frac{1}{2}\right)^{\frac{\rho}{\gamma}} \left[\kappa^{\frac{1}{2}} Y^{\frac{1+\rho}{2}} \right]^{\frac{1-\gamma}{\gamma}} \frac{E_{-1}(\kappa^{\frac{1}{\gamma}} Y^{\left[\frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma}\right]})}{E_{-1}(\kappa^{\frac{1-\gamma}{2\gamma}} Y^{\left[\frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma}\right]-\rho} \Pi^{-1})}, \text{(price)} \quad (64)$$

$$Y^{-\rho} = \beta(1+i)E(\Pi_{+1}^{-1} Y_{+1}^{-\rho}), \quad \text{(demand)} \quad (65)$$

and

$$I = I^* \beta^{-1} \kappa^{\lambda \kappa} \kappa_{-1}^{\lambda \kappa - 1} \Pi^{\lambda \pi}. \quad \text{(interest-rate rule)} \quad (66)$$

Equations (64)-(66) can be solved for output, inflation and the nominal interest rate, as shown in appendix C on page 22. The solutions for output and inflation are given by

$$Y = \Psi \kappa^{\psi} \quad \text{and} \quad \Pi = \Phi \kappa^{\phi \kappa} \kappa_{-1}^{\phi \kappa - 1},$$

where κ again is uniform between κ_L and κ_H , and the coefficients are as given in table 2.

Table 2: The sticky-price model solution	
$\psi = -\frac{\lambda_\kappa + \lambda_\pi \left\{ \frac{1-\gamma}{2\gamma} \right\} + \frac{\lambda_{\kappa-1}}{\lambda_\pi}}{\left[\rho + \frac{(1+\rho)(1-\gamma)\lambda_\pi}{2\gamma} \right]}$	(2.1)
$\phi_\kappa = \frac{1-\gamma}{\gamma 2} \left\{ 1 - \frac{(1+\rho) \left[\lambda_\kappa + \lambda_\pi \left\{ \frac{1-\gamma}{2\gamma} \right\} + \frac{\lambda_{\kappa-1}}{\lambda_\pi} \right]}{\left[\rho + \frac{1}{2\gamma} (1+\rho)(1-\gamma)\lambda_\pi \right]} \right\}$	(2.2)
$\phi_{\kappa-1} = -\frac{\lambda_{\kappa-1}}{\lambda_\pi}$	(2.3)
$\Phi = \left\{ \beta \kappa_L^{(-\lambda_\kappa - \phi_\kappa \lambda_\pi)} \right\} (\kappa_H^{1-\phi_\kappa - \rho\psi} - \kappa_L^{1-\phi_\kappa - \rho\psi}) \cdot (1 - \phi_\kappa - \rho\psi)^{-1} (\kappa_H - \kappa_L)^{-1}$	(2.4)
$\begin{aligned} \Psi &= (1-\gamma)^{-\frac{(1-\gamma)}{(1+\rho)}} \gamma^{-\frac{\gamma}{(1+\rho)}} 2^{\frac{\rho}{1+\rho}} \cdot \\ & \left(\kappa_H^{\psi \left\{ \frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma} \right\} + \frac{1+\gamma}{\gamma}} - \kappa_L^{\psi \left\{ \frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma} \right\} + \frac{1+\gamma}{\gamma}} \right)^{-\frac{1}{1+\rho} \gamma} \cdot \\ & \left\{ \psi \left\{ \frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma} \right\} + \frac{1+\gamma}{\gamma} \right\}^{\frac{\gamma}{1+\rho}} \cdot \\ & \left(\kappa_H^{1 + \frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa} - \kappa_L^{1 + \frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa} \right)^{\frac{\gamma}{1+\rho}} \cdot \\ & \left(1 + \frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa \right)^{\frac{-\gamma}{1+\rho}} \end{aligned}$	(2.5)
Memo: The equilibrium nominal interest rate: $1 + i = \left(\frac{\kappa}{\kappa_L} \right)^{(\lambda_\kappa + \phi_\kappa \lambda_\pi)} \geq 1$	

Note that equation (2.1) shows that output depends on current monetary policy, and monetary policy one period ahead, through λ_κ and $\lambda_{\kappa-1}$. In order to get some intuition for the constant terms Φ and Ψ , it is useful to note that with κ uniform between κ_L and κ_H , we have

$$E(\kappa^a) = \int_{\kappa_L}^{\kappa_H} \kappa^a \left(\frac{1}{\kappa_H - \kappa_L} \right) d\kappa = \left[\frac{1}{1+a} \kappa^{1+a} \frac{1}{\kappa_H - \kappa_L} \right]_{\kappa_L}^{\kappa_H} = \frac{1}{1+a} (\kappa_H^{1+a} - \kappa_L^{1+a}) \frac{1}{\kappa_H - \kappa_L}.$$

4.3 Inflation targeting in the sticky-price model

Authorities can achieve perfect stabilization of output (that is, output equal to potential, or flexible-price output) if they stabilize the inflation rate. Note that $\phi_\kappa = \phi_{\kappa-1} = 0$ is achieved by $\lambda_\kappa = \frac{\rho}{1+\rho}$ and $\lambda_{\kappa-1} = 0$ or by $\lambda_\pi \rightarrow \infty$. The additional ‘sticky-price part’ of the Ψ parameter drops out (compare the first line of (2.5) to equation (38)) and ψ reduces to $\frac{-1}{1+\rho}$, so that output is equal to the expression in (38). The reason why a constant inflation rate eliminates output gap distortions is that with a constant inflation rate there is no distortion of the relative price of sticky-price goods. This makes sure that the allocation of production and consumption across sectors is efficient.

With inflation fixed, output at its first best level and $\phi_\kappa = 0$ so that the nominal rate does not depend on λ_π , the λ_π is not identified under strict inflation targeting in the sticky price model either.

The nominal interest rate cannot be stable and equal to zero if the inflation rate is stabilized completely. But a zero nominal interest rate is required to achieve a first-best level of real money balances. The zero lower bound adds to the distortion of money demand when the inflation rate is stabilized, because not only will the nominal interest rate have to be variable. It has to be variable around a mean that is above zero and high enough to not make the nominal rate negative.

5 Optimal monetary policy in the sticky-price model

In this section, I show how the first-best level of both output and real money balances is attainable in the sticky-price model. When current prices are not free to adjust, but next period's prices are, any necessary movements in the inflation rate can happen via the future level of prices instead of via the current level of prices. If authorities respond to the current shock in the next period, and agents observe the current shock before they set next period's prices, the variation in the inflation rate comes as no surprise and is not costly. This feature of the model is due to synchronized contracts.

Recall that the first best level of output is given by equation (38), repeated here for convenience:

$$Y = \left(\frac{1}{(1-\gamma)^{1-\gamma}\gamma^\gamma} \right)^{\frac{1}{1+\rho}} \kappa^{-\frac{1}{1+\rho}} \left(\frac{1}{2} \right)^{-\frac{\rho}{1+\rho}}. \quad (67)$$

For the purpose of output stabilization, I am therefore looking for a rule that yields

$$\psi = -\frac{1}{1+\rho}.$$

With this ψ , Ψ will also be at its first-best level. From table 2, I know that I need

$$\psi = -\frac{\lambda_\kappa + \lambda_\pi \left\{ \frac{1-\gamma}{\gamma^2} \right\} + \frac{\lambda_{\kappa-1}}{\lambda_\pi}}{\left[\rho + \frac{(1+\rho)(1-\gamma)\lambda_\pi}{2\gamma} \right]} = -\frac{1}{1+\rho}. \quad (68)$$

(68) is satisfied if $\lambda_\pi \rightarrow \infty$ or if

$$\lambda_\kappa + \frac{\lambda_{\kappa-1}}{\lambda_\pi} = \frac{\rho}{1+\rho}. \quad (69)$$

$\phi_k = 0$ whenever (68) holds. With $\phi_k = 0$, movements in the price level are known when all price setters set their price and the relative price $\frac{P_s}{P_f}$ will be equal to its first-

best level $\left\{ \frac{\gamma}{1-\gamma} \right\}^{\frac{1}{2}}$. Intuitively, even if the inflation rate is not constant, the first-best level of output is reached if it is predictable.

Any combination of λ_κ , $\lambda_{\kappa-1}$ and λ_π that satisfies (69) or $\lambda_\pi \rightarrow \infty$ will ensure that (68) holds. But the case of

$$\frac{\lambda_{\kappa-1}}{\lambda_\pi} = \frac{\rho}{1+\rho} \text{ and } \lambda_\kappa = 0 \quad (70)$$

has particular interest. The reason is that the equilibrium nominal interest rate is given by

$$1+i = \left(\frac{\kappa}{\kappa_L}\right)^{\lambda_\kappa + \phi_\kappa \lambda_\pi}.$$

Since $\phi_\kappa = 0$ whenever (68) holds, the nominal interest rate is also stable and equal to zero when $\lambda_\kappa = 0$ and (68) holds and λ_π is bounded. Hence, as long as authorities can respond to the lagged supply shock but do not respond to the contemporaneous shock, they are able to implement the first-best solution for both output and real money balances. They are able to implement the first best solution by promising a time-varying inflation rate instead of keeping the inflation rate constant.

Note that authorities cannot postpone the effect of the κ shock on the inflation rate further by responding to shocks lagged more than one period, and still achieve the first-best solution. The reason is that authorities rely on the current variation in the inflation rate to create a real interest rate equal to the natural rate.

It might seem that authorities could infer the lagged supply shock from the first-best solution for the inflation rate, since the first best solution is $\Pi = \Phi \kappa_{\kappa-1}^{-\frac{\rho}{1+\rho}}$. One might therefore think that responding to a function of the inflation rate instead of the lagged shock directly could yield the first-best solution. However, the first-best inflation rate depends on the lagged shock only because monetary authorities respond to the lagged shock, as can be seen from the expression for $\phi_{\kappa-1}$ in table 2. If I eliminate $\lambda_{\kappa-1}$ from the interest-rate rule and let the authorities respond to the appropriate function of the inflation rate instead, all response parameters in the reaction function cancel out. Authorities are left with a rule that says they should peg the nominal rate at zero (or marginally higher). But then the model will not have a unique solution and the inflation rate and output will not be pinned down.

Also, one might think that authorities could implement the first-best solution by responding to the lagged inflation rate. This is not possible, however. This may be seen by considering the demand equation, repeated here:

$$Y^{-\rho} = \beta(1+i)E(\Pi_{+1}^{-1}Y_{+1}^{-\rho}).$$

I know that a first-best solution is characterized by output depending on κ only, and $i = 0$. Hence, *the only way the demand equation can hold in the first-best case is when the inflation rate depends on the lagged shock*. Furthermore, the inflation rate *cannot depend on the contemporaneous shock*: Whenever (68) holds so that $\psi = -\frac{1}{1+\rho}$, $\phi_\kappa = 0$. This means that I cannot have a first-best solution where the nominal interest rate responds to the lagged inflation rate as an indirect way of responding to

the lagged shock. In the first-best solution, the lagged inflation rate contains useless information about κ_{-2} and no information about κ_{-1} .

And again, the parameter λ_π is not identified by observing the equilibrium outcome, which is observationally equivalent to the first best equilibrium under flexible prices and does not depend on λ_π .

The optimal interest rate rule under optimal policy is given by

$$I = I^* \beta^{-1} \kappa_{-1}^{\frac{\rho}{1+\rho} \lambda_\pi} \Pi^{\lambda_\pi},$$

where, using $\lambda_\kappa = \phi_\kappa = 0$ in the first best solution and equation (C.11) on page 23,

$$I^* = \beta \Phi^{-\lambda_\pi},$$

Hence, the optimal interest rate rule is equal to

$$I = \Phi^{-\lambda_\pi} \kappa_{-1}^{\frac{\rho}{1+\rho} \lambda_\pi} \Pi^{\lambda_\pi}.$$

The intercept term $\Phi^{-\lambda_\pi} \kappa_{-1}^{\frac{\rho}{1+\rho} \lambda_\pi}$ depends the variance of κ through Φ and it is time-varying according to κ_{-1} .

6 Concluding remarks

With a Neo-Classical Phillips-Curve, the zero lower bound is not a constraint on optimal monetary policy. Optimal policy is history dependent in the sense that the natural real rate influences monetary policy with a lag, as is standard in the zero lower bound literature. Optimal policy here is not time-inconsistent. The intercept term in the interest rate reaction function that implements first best optimal policy has to be time-varying and it depends on the distribution of the natural real rate. The interest rate response to inflation that establishes determinacy is not identified under optimal policy.

Appendices

A The intratemporal cost minimization problem

This appendix presents demand functions derived from the agents' intratemporal cost minimization problem,

$$\text{Min} \left[P_s c_s + P_f c_f - \lambda \left[\frac{c_s^\gamma c_f^{1-\gamma}}{\gamma^\gamma (1-\gamma)^{1-\gamma}} - 1 \right] \right], \quad (\text{A.1})$$

where the agent minimizes with respect to c_s and c_f . P_s is the price index of sticky-price goods and P_f is the price index of flexible-price goods (time subscripts are suppressed here). I get

$$P_s = \lambda \gamma c_s^{-1} c, \quad P_f = \lambda (1 - \gamma) c_f^{-1} c. \quad (\text{A.2})$$

Define the minimum total expenditure required to obtain one unit of consumption c as P (the price index). P is then equal to λ . I use (A.2) to solve for $\lambda = P$ as a function of prices in the two sectors. First note that the constraint holds with equality in equilibrium, so $c = 1$. I then have

$$\lambda = P_s c_s \frac{1}{\gamma} = P_f c_f \frac{1}{1 - \gamma},$$

which after some manipulation gives the price index

$$P_s^\gamma P_f^{1-\gamma} = \lambda c^{-1} = \lambda \equiv P.$$

With $P = \lambda$, the demand functions can be written

$$c_s = \left(\frac{P_s}{P}\right)^{-1} \gamma c \quad \text{and} \quad c_f = \left(\frac{P_f}{P}\right)^{-1} (1 - \gamma) c.$$

From

$$c_{s,t} \equiv \left[\int_{j=0}^1 (c_{j,s,t})^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}, \quad c_{f,t} \equiv \left[\int_{i=0}^1 (c_{i,f,t})^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} \quad (\text{A.3})$$

I can derive demand for good j in terms of the relative price $\frac{p_{js}}{P_s}$;

$$\left(\frac{c_{js}}{c_s} \right) = \left(\frac{p_{js}}{P_s} \right)^{-\theta},$$

and similarly for flexible-price goods. This means that demand for an individual firms' goods in the sticky-price sector is

$$c_{js} = \left(\frac{p_{js}}{P_s} \right)^{-\theta} c_s = \left(\frac{p_{js}}{P_s} \right)^{-\theta} \left(\frac{P_s}{P} \right)^{-1} \gamma c. \quad (\text{A.4})$$

The corresponding relationship holds for flexible-price goods, but with $(1 - \gamma)$ instead of γ .

B The solution of the flexible-price model

The model is

$$\Pi = \Phi \kappa^{\phi_\kappa} \kappa_{-1}^{\phi_\kappa - 1}, \quad (\text{guess for } \Pi) \quad (\text{B.1})$$

$$\kappa^{\frac{\rho}{1+\rho}} = \beta(1+i)E(\Pi_{+1}^{-1} \kappa_{+1}^{\frac{\rho}{1+\rho}}), \quad (\text{demand}) \quad (\text{B.2})$$

and

$$1 + i = I^* \beta^{-1} \kappa^{\lambda_\kappa} \kappa_{-1}^{\lambda_{\kappa-1}} \Pi^{\lambda_\pi}. \quad (\text{interest-rate rule}) \quad (\text{B.3})$$

Use the interest-rate rule and the guess for Π in the demand equation and simplify to get

$$\kappa^{\frac{\rho}{1+\rho} - \lambda_\kappa - \phi_\kappa \lambda_\pi + \phi_{\kappa-1}} = I^* \Phi^{(\lambda_\pi - 1)} \kappa_{-1}^{\lambda_{\kappa-1} + \phi_{\kappa-1} \lambda_\pi} E(\kappa^{-\phi_\kappa + \frac{\rho}{1+\rho}}). \quad (\text{B.4})$$

In order for this equation to hold for all κ and κ_{-1} , it has to be the case that

$$\frac{\rho}{1+\rho} - \phi_\kappa \lambda_\pi - \lambda_\kappa + \phi_{\kappa-1} = 0 \quad \text{and} \quad \lambda_{\kappa-1} + \phi_{\kappa-1} \lambda_\pi = 0, \quad (\text{B.5})$$

which gives us equations (1.1) and (1.2) in table 1. With a uniform distribution of the shock¹⁸

$$E(\kappa^{-\phi_\kappa + \frac{\rho}{1+\rho}}) = \frac{1}{1 - \phi_\kappa + \frac{\rho}{1+\rho}} (\kappa_H^{1 - \phi_\kappa + \frac{\rho}{1+\rho}} - \kappa_L^{1 - \phi_\kappa + \frac{\rho}{1+\rho}}) \frac{1}{\kappa_H - \kappa_L}. \quad (\text{B.6})$$

Using (B.5) and (B.6), (B.4) can be rearranged to

$$1 = I^* \Phi^{(\lambda_\pi - 1)} \frac{1}{1 - \phi_\kappa + \frac{\rho}{1+\rho}} (\kappa_H^{1 - \phi_\kappa + \frac{\rho}{1+\rho}} - \kappa_L^{1 - \phi_\kappa + \frac{\rho}{1+\rho}}) \frac{1}{\kappa_H - \kappa_L}. \quad (\text{B.7})$$

I choose the parameter I^* in (B.3) so that the zero lower bound on the nominal interest rate is never violated. With optimal monetary policy it should hold with equality at least at one point, in order to minimize money-demand distortions. That is,

$$I = I^* \beta^{-1} \kappa^{\lambda_\kappa} \kappa_{-1}^{\lambda_{\kappa-1}} \Pi^{\lambda_\pi} \geq 1$$

must hold with equality at least at one point. Using $\phi_{\kappa-1} = -\frac{\lambda_{\kappa-1}}{\lambda_\pi}$ from (B.5) and substituting in for the solution for Π , I see that it must hold with equality either when $\kappa = \kappa_L$ or $\kappa = \kappa_H$.

$$I^* = \beta \kappa_L^{-\lambda_\kappa} \Phi^{-\lambda_\pi} \kappa_L^{-\phi_\kappa \lambda_\pi} \quad \text{when} \quad -(\lambda_\kappa + \phi_\kappa \lambda_\pi) < 0 \quad (\text{B.8})$$

or

$$I^* = \beta \kappa_H^{-\lambda_\kappa} \Phi^{-\lambda_\pi} \kappa_H^{-\phi_\kappa \lambda_\pi} \quad \text{when} \quad -(\lambda_\kappa + \phi_\kappa \lambda_\pi) > 0.$$

Given that the former condition applies¹⁹, I can solve for Φ by using (B.7) and (B.8):

$$\Phi = \beta \kappa_L^{-\lambda_\kappa - \phi_\kappa \lambda_\pi} \frac{1}{1 - \phi_\kappa + \frac{\rho}{1+\rho}} (\kappa_H^{1 - \phi_\kappa + \frac{\rho}{1+\rho}} - \kappa_L^{1 - \phi_\kappa + \frac{\rho}{1+\rho}}) \frac{1}{\kappa_H - \kappa_L},$$

¹⁸With κ uniform between κ_L and κ_H we have

$$E(\kappa^a) = \int_{\kappa_L}^{\kappa_H} \kappa^a \left(\frac{1}{\kappa_H - \kappa_L} \right) d\kappa = \left[\frac{1}{1+a} \kappa^{1+a} \frac{1}{\kappa_H - \kappa_L} \right]_{\kappa_L}^{\kappa_H} = \frac{1}{1+a} (\kappa_H^{1+a} - \kappa_L^{1+a}) \frac{1}{\kappa_H - \kappa_L}$$

The variance of κ is given by $\frac{1}{12}(\kappa_H - \kappa_L)$.

¹⁹We will see that in the first best solution, $\lambda_\kappa + \lambda_\pi \phi_\kappa = 0$. If authorities choose to respond to Π and κ , while $\lambda_{\kappa-1} = 0$, ϕ_κ will be non-negative. Hence, in all cases that I consider in this paper, the former case applies.

which is equation (1.3) in table 1. The equilibrium nominal interest rate is given by

$$1 + i = I^* \beta^{-1} \kappa^{\lambda_\kappa} \kappa_{-1}^{\lambda_{\kappa-1}} (\Phi^{\lambda_\pi} \kappa^{\phi_\kappa} \lambda_\pi \kappa_{-1}^{\phi_{\kappa-1}} \lambda_\pi) = \left(\frac{\kappa}{\kappa_L}\right)^{\lambda_\kappa + \phi_\kappa \lambda_\pi}, \quad (\text{B.9})$$

where I have used (B.8) and (B.5). I will want to adjust I^* to make sure that the nominal rate is always marginally higher than zero. In that case, I have $1 + i = \left(\frac{\kappa}{\kappa_L}\right)^{\lambda_\kappa + \phi_\kappa \lambda_\pi} (1 + \varepsilon)$.

C The solution of the sticky-price model

I will work with the system in logs. The price equation, interest-rate rule and demand equations are (small letters denoting logs of capital letters):

$$\begin{aligned} \pi = & \frac{1-\gamma}{\gamma} \left\{ \frac{1}{2} \ln \kappa + \frac{1+\rho}{2} y + \ln(1-\gamma) \right\} \\ & + \ln \gamma - \frac{\rho}{\gamma} \ln 2 + \ln \frac{E_{-1}(\kappa^{\frac{1}{\gamma}} Y^{[\frac{(1+\rho)(1-\gamma)+2\gamma]}{\gamma}]})}{E_{-1}(\kappa^{\frac{1-\gamma}{2\gamma}} Y^{[\frac{(1+\rho)(1-\gamma)+2\gamma]}{2\gamma}]^{-\rho} \Pi^{-1} \Pi_{-1}^{-1}) \Pi_{-1}} \end{aligned} \quad (\text{C.1})$$

$$i = i^* - \ln \beta + \lambda_\kappa \ln \kappa + \lambda_{\kappa-1} \ln \kappa_{-1} + \lambda_\pi \pi, \quad (\text{C.2})$$

and

$$-\rho y = \ln \beta + \ln I + \ln E(\Pi_{+1}^{-1} Y_{+1}^{-\rho}). \quad (\text{C.3})$$

I guess solutions for output and inflation in logs,

$$y = \ln \Psi + \psi \ln \kappa \quad (\text{C.4})$$

and

$$\pi = \ln \Phi + \phi_\kappa \ln \kappa + \phi_{\kappa-1} \ln \kappa_{-1}. \quad (\text{C.5})$$

First, substitute (C.2), (C.4) and (C.5) into (C.3) to get the following version of the demand equation:

$$\begin{aligned} -\rho[\ln \Psi + \psi \ln \kappa] = & i^* + \lambda_\kappa \ln \kappa + \lambda_{\kappa-1} \ln \kappa_{-1} \\ & + \lambda_\pi \phi_{\kappa-1} \ln \kappa_{-1} + \lambda_\pi [\ln \Phi + \phi_\kappa \ln \kappa] - \phi_{\kappa-1} \ln \kappa + \ln E(\Phi^{-1} \kappa_{+1}^{-\phi_\kappa} \Psi^{-\rho} \kappa_{+1}^{-\rho \psi}). \end{aligned} \quad (\text{C.5})$$

In order for this equation to hold for all κ , I need

$$-\rho \psi - \lambda_\kappa - \lambda_\pi \phi_\kappa + \phi_{\kappa-1} = 0 \quad (\text{C.6})$$

and also

$$\lambda_{\kappa-1} + \lambda_\pi \phi_{\kappa-1} = 0. \quad (\text{C.7})$$

Next, use (C.2), (C.4) and (C.5) in (C.1) to get the following version of the price equation

$$\begin{aligned} \phi_\kappa \ln \kappa = & \frac{1-\gamma}{\gamma} \left\{ \frac{1}{2} \ln \kappa + \frac{1+\rho}{2} (\ln \Psi + \psi \ln \kappa) + \ln(1-\gamma) \right\} \\ & + \ln \gamma - \left[\frac{\rho(1-\gamma)}{\gamma} + \rho \right] \ln 2 + \ln \frac{E_{-1}(\kappa^{\frac{1}{\gamma}} Y^{\frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma}})}{E_{-1}(\kappa^{\frac{1-\gamma}{2\gamma}} Y^{\frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma}-\rho} \kappa^{-\phi_\kappa})}. \end{aligned} \quad (\text{C.8})$$

In order for this equation to hold for all κ I need

$$\phi_\kappa = \frac{1-\gamma}{\gamma 2} + \frac{(1+\rho)(1-\gamma)}{2\gamma} \psi. \quad (\text{C.9})$$

I have solved for three of the parameters, equation (C.6), (C.7) and (C.9) are three equations in three unknown. These give equations (2.1)-(2.3) in table 2. As a consistency check, note that if $\gamma = 0$, so that all prices are flexible, I get $\psi = -\frac{1}{1+\rho}$ from (C.9), which means that output will be equal to potential (at least up to a scaling parameter, which is yet to be solved for). Furthermore, in this case I would get

$$\frac{\rho}{1+\rho} - \lambda_\kappa - \phi_{\kappa-1} = \lambda_\pi \phi_\kappa \Rightarrow \phi_\kappa = \frac{1}{\lambda_\pi} \left\{ \frac{\rho}{1+\rho} - \lambda_\kappa + \frac{\lambda_{\kappa-1}}{\lambda_\pi} \right\}$$

as in flexible-price case. I next need to derive

$$\ln E(\Phi^{-1} \kappa_{+1}^{-\phi_\kappa} \Psi^{-\rho} \kappa_{+1}^{-\rho\psi}) \quad (\text{C.10})$$

in order to get an equation in the unknown coefficients from the demand equation (C.5). As before I use a uniform distribution²⁰ of κ between κ_L and κ_H and get

$$\begin{aligned} \ln E(\Pi_{+1}^{-1} Y_{+1}^{-\rho}) = & -\ln \Phi - \rho \ln \Psi - \phi_{\kappa-1} \ln \kappa + \\ & \ln(\kappa_H^{1-\phi_\kappa-\rho\psi} - \kappa_L^{1-\phi_\kappa-\rho\psi}) - \ln(1 - \phi_\kappa - \rho\psi) - \ln(\kappa_H - \kappa_L). \end{aligned}$$

Substitute this into the demand equation (C.5), use (C.6), (C.7) and (C.9), rearrange and

$$\ln \Phi = \frac{1}{1-\lambda_\pi} \left\{ i^* + \ln(\kappa_H^{1-\phi_\kappa-\rho\psi} - \kappa_L^{1-\phi_\kappa-\rho\psi}) - \ln(1 - \phi_\kappa - \rho\psi) - \ln(\kappa_H - \kappa_L) \right\}.$$

As in the flexible-price case, I impose the constraint that $i^* = 0$ when the supply shock takes on the minimum value, so that²¹

$$\ln(1 + i^*) = \ln \beta - \lambda_\kappa \ln \kappa_L - \lambda_\pi \ln \Phi - \phi_\kappa \lambda_\pi \ln \kappa_L, \quad (\text{C.11})$$

which means that

$$\begin{aligned} \ln \Phi = & \frac{1}{1-\lambda_\pi} \left\{ \ln \beta - \lambda_\kappa \ln \kappa_L - \lambda_\pi \ln \Phi - \phi_\kappa \lambda_\pi \ln \kappa_L \right. \\ & \left. + \ln(\kappa_H^{1-\phi_\kappa-\rho\psi} - \kappa_L^{1-\phi_\kappa-\rho\psi}) - \ln(1 - \phi_\kappa - \rho\psi) - \ln(\kappa_H - \kappa_L) \right\}. \end{aligned}$$

²⁰See footnote 18.

²¹In order to ensure a unique equilibrium I will want to add a constant term $\ln(1 + \varepsilon)$, so that the nominal rate becomes strictly positive.

Simplifying and expressing this in levels, I get

$$\Phi = \{\beta \kappa_L^{-\phi_\kappa \lambda_\pi - \lambda_\kappa}\} (\kappa_H^{1-\phi_\kappa - \rho\psi} - \kappa_L^{1-\phi_\kappa - \rho\psi}) (1 - \phi_\kappa - \rho\psi)^{-1} (\kappa_H - \kappa_L)^{-1}.$$

This is equation (2.4) in table 2. For the purpose of the price equation (C.8), I need

$$\begin{aligned} & \ln E_{-1}(\kappa^{\frac{1}{\gamma}} \{\Psi \kappa^\psi\}^{[\frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma}]}) - \ln E_{-1}(\kappa^{\frac{1-\gamma}{2\gamma}} \{\Psi \kappa^\psi\}^{[\frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma}] - \rho} \kappa^{-\phi_\kappa}) \quad (\text{C.12}) \\ &= \left\{ \left[\frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} \right] + \rho \right\} \ln \Psi \\ &+ \ln(\kappa_H^{1+\psi \frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma} + \frac{1}{\gamma}} - \kappa_L^{1+\psi \frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma} + \frac{1}{\gamma}}) \\ &- \ln\left(1 + \psi \frac{\{(1+\rho)(1-\gamma)+2\gamma\}}{\gamma} + \frac{1}{\gamma}\right) \\ &- \ln(\kappa_H^{1+\frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa} - \kappa_L^{1+\frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa}) \\ &+ \ln\left(1 + \frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa\right). \end{aligned}$$

This expression must be substituted into (C.8). In addition, use the solutions already obtained for ϕ_κ , $\phi_{\kappa-1}$ and ψ , and (C.8) becomes

$$\begin{aligned} \ln \Psi &= -\frac{1}{(1+\rho)} [(1-\gamma) \ln(1-\gamma) + \gamma \ln \gamma - \rho \ln 2 + \quad (\text{C.13}) \\ &+ \gamma \ln(\kappa_H^{1+\psi \frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma} + \frac{1}{\gamma}} - \kappa_L^{1+\psi \frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma} + \frac{1}{\gamma}}) \\ &- \gamma \ln\left(1 + \psi \frac{\{(1+\rho)(1-\gamma)+2\gamma\}}{\gamma} + \frac{1}{\gamma}\right) \\ &- \gamma \ln(\kappa_H^{1+\frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa} - \kappa_L^{1+\frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa}) \\ &+ \gamma \ln\left(1 + \frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa\right)]. \end{aligned}$$

Note that the constant term includes the term that I would have in the flexible-price case (the right hand side of the first line of (C.13)), but adds on a term that depends on the distribution of κ . In levels, (C.13) becomes

$$\begin{aligned} \Psi &= (1-\gamma)^{-\frac{(1-\gamma)}{(1+\rho)}} \gamma^{-\frac{\gamma}{(1+\rho)}} 2^{\frac{\rho}{1+\rho}} \cdot \\ &(\kappa_H^{\psi \frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma} + \frac{1+\gamma}{\gamma}} - \kappa_L^{\psi \frac{(1+\rho)(1-\gamma)+2\gamma}{\gamma} + \frac{1+\gamma}{\gamma}})^{-\frac{1}{1+\rho}} \gamma \\ &\cdot \left\{ \psi \frac{\{(1+\rho)(1-\gamma)+2\gamma\}}{\gamma} + \frac{1+\gamma}{\gamma} \right\}^{\frac{\gamma}{1+\rho}} \cdot \\ &(\kappa_H^{1+\frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa} - \kappa_L^{1+\frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa})^{\frac{\gamma}{1+\rho}} \cdot \\ &\left(1 + \frac{1+\gamma}{2\gamma} + \psi \frac{(1+\rho)(1-\gamma)+2\gamma}{2\gamma} - \psi\rho - \phi_\kappa\right)^{\frac{-\gamma}{1+\rho}}. \end{aligned}$$

This is equation (2.5) in table 2. The expression for the equilibrium nominal interest rate is as in the flexible-price case.

D Solution when price setters in the sticky-price sector have no information about the current shock

Taking expectations two periods in advance in the first order condition for optimal price setting in the sticky-price sector (equation (22)) but otherwise doing the same substitutions implies that the model becomes

$$\Pi = \left[\frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}} \Pi \right]^{1-\gamma} \left[\frac{E_{-2}(\kappa Y_s^2)}{E_{-2}(Y_s(\frac{1}{2}Y)^{-\rho} \Pi^{-1} \Pi_{-1}^{-1}) \Pi_{-1}} \right]^\gamma, \quad (\text{price equation}) \quad (\text{E.1})$$

$$\left(\frac{1}{2}Y\right)^{-\rho} = \beta(1+i)E(\Pi_{+1}^{-1} \left\{ \frac{1}{2}Y_{+1} \right\}^{-\rho}), \quad (\text{demand}) \quad (\text{E.2})$$

$$Y_f = \left(\frac{\kappa Y_f}{(\frac{1}{2}Y)^{-\rho}}\right)^{-1} (1-\gamma)Y, \quad (\text{flex-price output}) \quad (\text{E.3})$$

and

$$Y_s = \left[\left(\frac{E_{-2}\{\kappa Y_s^2\}}{E_{-2}\{Y_s(\frac{1}{2}Y)^{-\rho} \Pi^{-1} \Pi_{-1}^{-1}\} \Pi_{-1}} \right)^{-1} \Pi \gamma Y \right]. \quad (\text{sticky-price output}) \quad (\text{E.4})$$

Substituting out for Y_f and Y_s in the same way as for the model in the main text and taking logs, the system becomes

$$\pi = \frac{1-\gamma}{\gamma} \left\{ \frac{1}{2} \ln \kappa + \frac{1+\rho}{2} y + \ln(1-\gamma) \right\} \quad (\text{E.5})$$

$$+ \ln \gamma - \left[\frac{\rho}{\gamma} \right] \ln 2 + \ln \frac{E_{-2}(\kappa^\gamma Y^{[\frac{(1+\rho)(1-\gamma)+2\gamma]}{\gamma}]})}{E_{-2}(\kappa^{\frac{1-\gamma}{2\gamma}} Y^{[\frac{(1+\rho)(1-\gamma)+2\gamma]}{2\gamma}]^{-\rho} \Pi^{-1} \Pi_{-1}^{-1}) \Pi_{-1}},$$

$$i = i^* - \ln \beta + \lambda_\kappa \ln \kappa + \lambda_{\kappa-1} \ln \kappa_{-1} + \lambda_{\kappa-2} \ln \kappa_{-2} + \lambda_\pi \pi, \quad (\text{E.6})$$

and

$$-\rho y = \ln \beta + \ln I + \ln E(\Pi_{+1}^{-1} Y_{+1}^{-\rho}). \quad (\text{E.7})$$

I guess solutions for output and inflation in logs²²,

$$y = \ln \Psi + \psi \ln \kappa. \quad (\text{E.8})$$

and

$$\pi = \ln \Phi + \phi_\kappa \ln \kappa + \phi_{\kappa-1} \ln \kappa_{-1} + \phi_{\kappa-2} \ln \kappa_{-2}. \quad (\text{E.9})$$

First, substitute equations (E.6), (E.8) and (E.9) into equation (E.7). I get

$$\begin{aligned} -\rho[\psi \ln \kappa] &= \ln \beta + [i^* - \ln \beta + \lambda_\kappa \ln \kappa + \lambda_{\kappa-1} \ln \kappa_{-1} \\ &\quad + \lambda_{\kappa-2} \ln \kappa_{-2} + \lambda_\pi \{ \ln \Phi + \phi_\kappa \ln \kappa + \phi_{\kappa-1} \ln \kappa_{-1} \\ &\quad + \phi_{\kappa-2} \ln \kappa_{-2} \}] - \phi_{\kappa-2} \kappa_{-1} - \phi_{\kappa-1} \ln \kappa + \ln E(\Phi^{-1} \kappa_{+1}^{-\phi_\kappa} \kappa_{+1}^{-\rho \psi}). \end{aligned}$$

²²I have included response to the two-period lags shock in the interest rate rule and a term including κ_{-2} in the guess for the inflation rate. This is done in order to make the model more general.

In order for this equation to hold for all κ, κ_{-1} and κ_{-2} I need

$$-\rho\psi = \lambda_\kappa + \lambda_\pi\phi_\kappa - \phi_{\kappa-1}. \quad (\text{E.10})$$

Furthermore,

$$\lambda_{\kappa-1} + \lambda_\pi\phi_{\kappa-1} - \phi_{\kappa-2} = 0, \quad (\text{E.11})$$

and

$$\lambda_{\kappa-2} + \lambda_\pi\phi_{\kappa-2} = 0. \quad (\text{E.12})$$

Next, substitute equations (E.8) and (E.9) into (E.5) to get:

$$\begin{aligned} \phi_\kappa \ln \kappa + \phi_{\kappa-1} \ln \kappa_{-1} &= \frac{1-\gamma}{\gamma} \left\{ \frac{1}{2} \ln \kappa + \frac{1+\rho}{2} [\ln \Psi + \psi \ln \kappa] + \ln(1-\gamma) \right\} \\ &+ \ln \gamma - \left[\frac{\rho}{\gamma} \ln 2 + \left[\frac{(1+\rho)(1-\gamma) + 2\gamma}{2\gamma} \right] + \rho \right] \ln \Psi \\ &+ \ln \frac{E_{-2}(\kappa^{\frac{1}{\gamma}} \kappa^{\psi \left[\frac{(1+\rho)(1-\gamma) + 2\gamma}{\gamma} \right]})}{E_{-2}(\kappa^{\frac{1-\gamma}{2\gamma}} \kappa^{\psi \left[\frac{(1+\rho)(1-\gamma) + 2\gamma}{2\gamma} \right] - \psi \rho} \kappa^{-\phi_\kappa} \kappa_{-1}^{-\phi_{\kappa-1} - \phi_\kappa})} - \phi_\kappa \ln \kappa_{\kappa-1}. \end{aligned} \quad (\text{E.13})$$

I now see that I need

$$\phi_\kappa = \frac{1-\gamma}{2\gamma} + \frac{(1-\gamma)(1+\rho)\psi}{2\gamma} \text{ and } \phi_{\kappa-1} = -\phi_\kappa \quad (\text{E.14})$$

in order for this to hold for all κ, κ_{-1} . If the κ_{-1} shock had been known when expectations were made, there would have been no restriction on $\phi_{\kappa-1}$ resulting from this equation. The two equations involving ψ are $-\rho\psi = \lambda_\kappa + \lambda_\pi\phi_\kappa - \phi_{\kappa-1}$ and $\phi_\kappa = \frac{1-\gamma}{2\gamma} + \frac{(1-\gamma)(1+\rho)\psi}{2\gamma}$. These are the same as before, except that $\phi_{\kappa-1}$ is restricted to be equal to $-\phi_\kappa$, which means that

$$\psi = -\frac{\lambda_\kappa + (\lambda_\pi + 1) \left\{ \frac{1-\gamma}{\gamma^2} \right\}}{[\rho + (\lambda_\pi + 1) \left(\frac{(1+\rho)(1-\gamma)}{2\gamma} \right)]}. \quad (\text{E.15})$$

The constant terms are equal to the case in the main text. To see this, note that since $\phi_{\kappa-1} = -\phi_\kappa$, the term involving expectations of κ_{-1} drops out from inside the expectation operator in equation (E.13). The other constant terms are as before. Terms involving $\lambda_{\kappa-1}, \lambda_{\kappa-2}$ and $\phi_{\kappa-2}$ cancel out and the nominal interest rate is $i = i^* - \ln \beta + \lambda_\kappa \ln \kappa + \lambda_\pi \{ \ln \Phi + \phi_\kappa \ln \kappa \}$. I ensure that the nominal rate is never negative but as low as possible with the same restriction on i^* as before. The equilibrium nominal interest rate is

$$i = (\lambda_\kappa + \lambda_\pi\phi_\kappa) \ln \kappa - (\lambda_\kappa + \lambda_\pi\phi_\kappa) \ln \kappa_L, \quad (\text{E.16})$$

as in the model of the main text. Again, the first-best solution for output is achieved by $\lambda_\kappa = \frac{\rho}{1+\rho}$ or by $\lambda_\pi \rightarrow \infty$. But now, $\lambda_{\kappa-1}$ cannot be used as an alternative way to achieve the first-best solution for output. Since $\phi_\kappa = 0$ in the first-best solution for output, $\phi_{\kappa-1} = 0$ as well. In order for the nominal rate to be stable I need $\lambda_\kappa + \lambda_\pi\phi_\kappa = 0$ exactly as in the case with one period preset prices. But given the first-best solution for output, $\psi = -\frac{1}{1+\rho}$, I have (using E.10)

$$\lambda_\kappa + \lambda_\pi\phi_\kappa = \frac{\rho}{1+\rho}.$$

Hence the first-best solutions for output and real money balances are not simultaneously attainable. The inflation rate may depend on κ_{-2} if $\lambda_{\kappa_{-2}} \neq 0$ as seen from equation E.12. But that does not help achieve the first-best solution.

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