An Efficient Method of Computing Higher Order Bond Price Perturbation Approximations

Martin M. Andreasen*
Bank of England
CREATES

Pawel Zabczyk
Bank of England
LSE, Dept. of Economics

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Abstract

This paper develops a fast method of computing arbitrary order perturbation approximations to bond prices in DSGE models. The procedure is implemented to third order where it can shorten the approximation process by more than one hundred times. In a consumption-based endowment model with habits, it is further shown that a third order perturbation solution is more accurate than the log-normal method and a procedure using consol bonds. In addition, we present MATLAB codes that implement the suggested method to third order.

Keywords: Perturbation method, DSGE models, Habit model, Higher order approximation.

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*Corresponding author: martin.andreasen@bankofengland.co.uk. Bank of England, Threadneedle Street, London EC2R 8AH.

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1 Introduction

Dynamic Stochastic General Equilibrium (DSGE) models are frequently used to analyze macroeconomic and yield curve dynamics jointly.[1] Since closed-form solutions to these models are in general unavailable, both the functions capturing state dynamics as well as those mapping state variables into asset prices need to be approximated. It is well established that the standard log-linear method is inadequate in an asset pricing context as it restricts risk premia to be zero, which goes counter to existing evidence (see e.g. Campbell & Shiller (1991) or Cochrane & Piazzesi (2005)). The most widely used alternative is to resort to more accurate higher order perturbation methods (see also Arouba, Fernández-Villaverde & Rubio-Ramírez (2005) and Caldara, Fernandez-Villaverde, Rubio-Ramirez & Yao (2009)).[2] But even these can become impractical when the approximated model features a yield curve.

To understand why, consider a quarterly DSGE model with n state variables. Assume further that we are interested in computing the 10 year yield from the price of a zero-coupon bond with the same maturity. This bond price is a function of n state variables and to approximate it to third order – i.e. by a third order polynomial – would require computing

$$\underbrace{1}_{0^{\text{th}}\text{ Order Terms}} + \underbrace{n}_{1^{\text{st}}\text{ Order Terms}} + \underbrace{n \cdot \frac{n+1}{2}}_{2^{\text{nd}}\text{ Order Terms}} + \underbrace{n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}}_{3^{\text{rd}}\text{ Order Terms}}$$

i.e. a total of $n \cdot (n+1)/2 \cdot (n+2)/3$ distinct coefficients. Typically, the 10 year bond price is computed recursively along with all 40 intermediate bond prices.[3] Accordingly, for n equal to 5, 10, or 15 the yield curve introduces respectively 2 240, 11 440, or 32 640 additional coefficients to be simultaneously computed. This can either make the problem too large to solve using standard packages or significantly increase the time required to approximate solutions.[4] While the deterioration in performance might

[1] Examples include Bekaert, Cho & Moreno (2005), Wu (2006), De Paoli, Scott & Weeken (2007), Hordahl, Tristani & Vestin (2008), Rudebusch & Swanson (2008), and Rudebusch & Swanson (2009).

[2] Other possibilities include value function iteration, finite elements, and Chebyshev polynomials, but these are typically considered unfeasible for medium-scale DSGE models.

[3] Alternative, non-recursive methods involve creating many auxiliary variables which similarly complicates the approximation problem.

[4] These packages include *Dynare*, *Dynare++,* and *Perturbation AIM* (see Kamenik (2005) and Swanson, Anderson & Levin (2005), respectively) and the set of routines accompanying Schmitt-Grohé & Uribe (2004).

be tolerable if the model needs to be solved once, it has the potential to make estimation or sensitivity analysis infeasible as both of them rely on repeated approximations.

The contribution of this paper is to propose a method of reducing the computational burden when approximating bond prices to arbitrary order. MATLAB codes that implement the suggested method to third order are also provided. We focus on the standard case in which bond prices with maturities beyond one period do not affect the rest of the economy, but alternatives are also considered.[5] The solution we advocate splits the perturbation problem into two steps. In the first step, standard solution packages can be used to approximate the solution to a DSGE model *without* bond prices of maturity greater than one. The second step perturbs the fundamental pricing equation for bond prices up to the same order. We then exploit the information from the first step to recursively solve for the coefficients of bond prices, significantly speeding up the approximation process. On account of this structure, we refer to our method as 'perturbation-on-perturbation' (POP). It is important to emphasize that the POP method computes *exactly* the same expressions for bond prices as the standard 'one-step' perturbation.

Our proposed method is closest to the one proposed in Hordahl et al. (2008).[6] They first approximate a solution to the part of their DSGE model without bond prices to second order. Afterwards, all bond prices are solved for using the fundamental asset pricing equations and the first approximation. Notably, we extend their work along three dimensions. Firstly, we go beyond second order and provide third order accurate formulas for bond prices. As mentioned in Schmitt-Grohé & Uribe (2004), third order terms are of economic significance as they allow for time variation in risk premia. Secondly, we allow for more general transformations of variables in the model than the 'log' specification considered in Hordahl et al. (2008). Thirdly, we consider a slightly more general setup than in Hordahl et al. (2008), as we do not introduce restrictions on the functional form of the stochastic discount factor.

A simulation study is then used to document the reduction in computational burden achieved by using the POP method instead of standard one-step perturbation. For the DSGE models of Rudebusch &

[5] Expressed alternatively, the assumption we rely on is that the model is such that prices of all bonds exceeding one period only appear in consumption-Euler equations.

[6] Binsbergen, Fernandez-Villaverde, Koijen & Rubio-Ramirez (2010) independently apply a related method to compute interest rates in a version of the neoclassical growth model. The method and formulas we provide are not model specific and our approach nests their procedure.

Swanson (2008) and De Paoli et al. (2007) speed gains vary from between 14 and 23 times for the case of a 10 year yield curve to between 61 and 139 times for a 20 year yield curve. As demonstrated in Andreasen (2010*a*), the speed gains involved are sufficient to make estimation of medium-scale DSGE models with a whole yield curve approximated to third order feasible.

We then assess accuracy of the POP method using closed-form solutions for bond prices in a consumption based model with habits (Zabczyk (2010)). Broadly in line with Arouba et al. (2005) and Caldara et al. (2009) we find that third order approximations outperform alternatives including the log-normal approach (Jermann (1998), Doh (2007)) and the method using consol bonds proposed in Rudebusch & Swanson (2008).

The remainder of this paper is organized as follows: section 2 describes the POP method, section 3 documents the gains in speed (at third order), accuracy is assessed in section 4, and section 5 concludes.

2 The POP method of computing bond prices

This section presents the POP method of approximating bond prices. For parsimony, we adopt the same framework as in Schmitt-Grohé & Uribe (2004).[7] We further assume that the model can be split into two parts. The first part contains all equations in which bond prices beyond one period maturity do *not* appear. The second part consists entirely of Euler equations for the remaining bond prices.[8] Hence, let $\mathbf{y}_t$ denote the $n_y \times 1$ vector of all non-predetermined variables *except* bond prices with a maturity exceeding one period, and let $\mathbf{x}_t$ be the $n_x \times 1$ vector of predetermined state variables. As in Schmitt-Grohé & Uribe (2004), the solution can be written as

$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_t, \sigma\right) \tag{1}$$

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t, \sigma\right) + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \tag{2}$$

[7] Extensions to the more general case in which shocks do not necessarily enter additively, as in *Dynare* (Kamenik (2005)) or *Perturbation AIM* (Swanson et al. (2005)), are straightforward and dealt with in the appendix.

[8] This structure is standard and all macro-finance models listed in footnote 1 satisfy this assumption.

where $\boldsymbol{\epsilon}_{t+1} \sim \mathcal{IID}\left(\mathbf{0}, \mathbf{I}\right)$ is a vector of n_ϵ innovations, $\boldsymbol{\eta}$ denotes the square root of their covariance matrix, and σ is the perturbation parameter. In the first step of the POP method the solution (1) - (2) is approximated to N-th order using standard perturbation methods.

Let $P^{t,\,k}$ denote the price in period t of a zero-coupon bond maturing in k periods with a face value of one. The price of this bond satisfies the fundamental pricing equation (see Cochrane (2001))

$$P^{t,\,k} = E_t\left[\mathcal{M} \times P^{t+1,\,k-1}\right]$$

for $k = 1, 2, ..., K$ where $\mathcal{M}$ is the stochastic discount factor. In many applications focus is on logarithms of prices rather than their levels. To accommodate this possibility we could rewrite the equation above as

$$\exp\left(\hat{p}^{t,\,k}\right) = E_t\left[\mathcal{M} \times \exp\left(\hat{p}^{t+1,\,k-1}\right)\right]$$

where $\hat{p}^{t,\,k} \equiv \log(P^{t,\,k})$. More generally, since other transformations might be useful when solving DSGE models (see for example Fernandez-Villaverde & Rubio-Ramirez (2006)), we introduce an invertible transformation function $R\left(\cdot\right) \in \mathcal{C}^N$ and denote $p^{t,\,k} \equiv R^{-1}(P^{t,\,k})$. The pricing equation can then be written as

$$R\left(p^{t,\,k}\right) = E_t\left[\mathcal{M} \times R\left(p^{t+1,\,k-1}\right)\right]. \tag{3}$$

Setting $R(x) = x$ gives the original 'levels' specification, while letting $R(x) = \exp(x)$ corresponds to the case of a log-transformation. The gross yield-to-maturity in period t of a k period bond $YTM^{t,k}$ can still be computed and is given by $YTM^{t,k} = (P^{t,k})^{-1/k} = R(p^{t,\,k})^{-1/k}$.

To compute perturbation approximations to $p^{t,\,k}$ we exploit two facts. Firstly, the the functional form of the stochastic discount factor $\mathcal{M}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t)$ is known.[9] Secondly, since any bond price is non-predetermined, it is a function of $\mathbf{x}_t$ and σ. We denote this function by $p^k\left(\mathbf{x}_t, \sigma\right)$ where k denotes the maturity of the bond. Where no ambiguity can arise, we omit the function arguments and simply

[9] We assume that the variables in the first block of the model, i.e. $\mathbf{x}$ and $\mathbf{y}$, have also been transformed using $R(\cdot)$. Accordingly, $\mathcal{M}$ and all its derivatives are known functions of the *transformed* variables. For example, for CRRA utility and $R(x) = \exp(x)$ we would have $\mathcal{M}(c_{t+1}, c_t) = \beta \exp(-\gamma\, c_{t+1}) / \exp(-\gamma\, c_t)$.

write $p^{t,k}$. Using these insights and substituting (1) and (2) into (3), we then define

$$F^k(\mathbf{x},\sigma) := E\Big\{R\left(p^k(\mathbf{x},\sigma)\right) - R\left(p^{k-1}\left(\mathbf{h}(\mathbf{x},\sigma)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\sigma\right)\right)$$
$$\times\,\mathcal{M}\left(\mathbf{g}\left(\mathbf{h}(\mathbf{x},\sigma)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\sigma\right),\mathbf{g}(\mathbf{x},\sigma),\mathbf{h}(\mathbf{x},\sigma)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1},\mathbf{x}\right)\Big\}\quad \text{for } k=1,2,...,K. \qquad (4)$$

It follows by construction that $F^k(\mathbf{x},\sigma)\equiv 0$ for all values of $\mathbf{x}$ and σ. Clearly, this implies that all derivatives of $F^k(\mathbf{x},\sigma)$ must also equal zero, i.e.

$$F^k_{\mathbf{x}^i\sigma^j}(\mathbf{x},\sigma)=0 \qquad \forall \mathbf{x},\sigma,i,j \qquad (5)$$

where $F^k_{\mathbf{x}^i\sigma^j}(\mathbf{x},\sigma)$ denotes the derivative of F^k with respect to $\mathbf{x}$ taken i times and with respect to σ taken j times. In the following subsections, we show how (5) together with the output from the first perturbation step (1) and (2) can be used to find derivatives of $p^k(\mathbf{x},\sigma)$ of order up to N evaluated at the deterministic steady state. These derivatives suffice to construct an N-th order perturbation approximation to $p^k(\mathbf{x},\sigma)$ around the deterministic steady state.

2.1 Notation

To make the subsequent formulas more transparent, we adopt the convention that indices α and γ relate to elements of $\mathbf{x}$, while β and ϕ correspond to elements of $\mathbf{y}$ and $\boldsymbol{\epsilon}$, respectively. Furthermore, subscripts on these indices will capture the sequence in which derivatives are being taken. For example, α_1 corresponds to the first time a function is differentiated with respect to $\mathbf{x}$, while α_2 is used when differentiating with respect to $\mathbf{x}$ for the second time.

In most of the subsequent derivations we follow Schmitt-Grohé & Uribe (2004) and use the Einstein summation notation. In particular, $[p^k_{\mathbf{x}}]_{\gamma_1}$ denotes the γ_1-th element of the $1\times n_x$ vector of derivatives of p^k with respect to $\mathbf{x}$. Similarly, the derivative of $\mathbf{h}$ with respect to $\mathbf{x}$ is an $n_x\times n_x$ matrix and $[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1}$ is the element of this matrix located at the intersection of the γ_1-th row and the α_1-th column. Also, $\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} = \sum_{\gamma_1=1}^{n_x}(\partial p^{k-1}/\partial \mathbf{x}_{\gamma_1})(\partial \mathbf{h}^{\gamma_1}/\partial \mathbf{x}_{\alpha_1})$ while $\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} = \sum_{\gamma_1=1}^{n_x}\sum_{\gamma_2=1}^{n_x}(\partial^2 p^{k-1}/\partial \mathbf{x}_{\gamma_1}\partial \mathbf{x}_{\gamma_2})(\partial \mathbf{h}^{\gamma_2}/\partial \mathbf{x}_{\alpha_2})(\partial \mathbf{h}^{\gamma_1}/\partial \mathbf{x}_{\alpha_1})$ where, for instance, $\mathbf{h}^{\gamma_1}$ denotes the γ_1-th function of mapping $\mathbf{h}$ and $\mathbf{x}_{\alpha_1}$ is the α_1-th element of vector $\mathbf{x}$.

For parsimony, we also use superscripts t and $t+1$ on functions p^k, $\mathbf{h}$, $\mathbf{g}$, and their derivatives to indicate the arguments at which they are evaluated. When these superscripts are omitted, functions are evaluated at the deterministic steady state, i.e. for $(\mathbf{x}, \sigma) = (\mathbf{x}_{ss}, 0)$. For example, for $f \in \{p^k, \mathbf{g}, \mathbf{h}\}$

$$f^t := f(\mathbf{x}_t, \sigma) \qquad f^{t+1} := f(\mathbf{x}_{t+1}, \sigma) \qquad f := f(\mathbf{x}_{ss}, 0)$$

$$f^t_{\mathbf{x}} := (\partial f / \partial \mathbf{x})|_{(\mathbf{x}_t, \sigma)} \qquad f^{t+1}_{\mathbf{x}} := (\partial f / \partial \mathbf{x})|_{(\mathbf{x}_{t+1}, \sigma)} \qquad f_{\mathbf{x}} := (\partial f / \partial \mathbf{x})|_{(\mathbf{x}_{ss}, 0)}.$$

2.2 Finding the first order derivatives

To find the first order derivatives of $p^k(\mathbf{x}, \sigma)$ with respect to $\mathbf{x}$, we start by differentiating $F^k(\mathbf{x}, \sigma)$ with respect to $\mathbf{x}$. Exploiting (5) we rewrite $\left[F^k_{\mathbf{x}}(\mathbf{x}_t, \sigma)\right]_{\alpha_1} = 0$ as

$$R_p\left(p^k\right)\left[p^{t,k}_{\mathbf{x}}\right]_{\alpha_1} - [\mathcal{M}_{\mathbf{x}}]_{\alpha_1} R\left(p^{t+1,k-1}\right) - \mathcal{M} R_p\left(p^{t+1,k-1}\right)\left[p^{t+1,k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}^t_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} = 0 \tag{6}$$

for $\alpha_1, \gamma_1 \in \{1, 2, ..., n_x\}$. Evaluating (6) in the deterministic steady state gives a set of equations which determine $\left[p^k_{\mathbf{x}}\right]_{\alpha_1}$ for $\alpha_1 = 1, 2, ..., n_x$ and $k = 1, 2, ..., K$. Given the output from the first perturbation step, we now show how these derivatives can be recursively solved for.

To show this and to establish the recursive argument, consider first the price of a bond with one period to maturity. The price of a maturing bond is one for all values of $(\mathbf{x}, \sigma)$, and all of its derivatives are therefore equal zero, i.e. $p^{t+1,0}_{\mathbf{x}} = 0$. Accordingly, equation (6) evaluated at the steady state and for $k = 1$ simplifies to

$$R_p\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_1} = [\mathcal{M}_{\mathbf{x}}]_{\alpha_1}, \tag{7}$$

where we use $R\left(p^0\right) = P^0 = 1$. The value of $R_p\left(p^1\right)$ can easily be computed from its known functional form and the steady state value of p^1. Further, the value of $[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ evaluated at the steady state can readily be found by differentiating $\mathcal{M}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t)$ and exploiting equations (1) and (2)

$$[\mathcal{M}_{\mathbf{x}}]_{\alpha_1} = \left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} + \left[\mathcal{M}_{\mathbf{y}_t}\right]_{\beta_1}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\alpha_1} + \left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]^{\gamma_1}_{\alpha_1} + [\mathcal{M}_{\mathbf{x}_t}]_{\alpha_1}, \tag{8}$$

along with the fact that $\mathbf{g_x}$, $\mathbf{h_x}$, and all the derivatives of $\mathcal{M}$ are known in the deterministic steady state. An alternative and slightly easier way of obtaining $[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ is to report it in the first perturbation step. However, this often comes at the cost of introducing extra variables into the state vector which slows down the first step of the POP method. Once the scalar $R_p\left(p^1\right)$ and $[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ have been computed, the derivatives $\left[p_{\mathbf{x}}^1\right]_{\alpha_1}$ are immediately given by (7).

Given that we know $\left[p_{\mathbf{x}}^1\right]_{\alpha_1}$, we can then compute the first order terms for the remaining maturities directly from (6). To do that we evaluate (6) in the deterministic steady state. Using $\mathcal{M} = R\left(p^1\right)$ and substituting out for $[\mathcal{M}_{\mathbf{x}}]_{\alpha_1}$ from (7), we obtain the following system of equations for $p_{\mathbf{x}}^k$

$$R_p\left(p^k\right)\left[p_{\mathbf{x}}^k\right]_{\alpha_1} = \left[p_{\mathbf{x}}^1\right]_{\alpha_1} R_p\left(p^1\right) R\left(p^{k-1}\right) + R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h_x}]_{\alpha_1}^{\gamma_1} \tag{9}$$

for $k = 2, 3, ..., K$ and $\alpha_1, \gamma_1 \in \{1, 2, \dots, n_x\}$. Again, $R_p\left(p^k\right)$ is a scalar and all the terms on the right hand side are known, which makes it straightforward to solve for $\left[p_{\mathbf{x}}^k\right]_{\alpha_1}$. In the special case of a log-transformation, the expression in (9) simplifies to

$$\mathbf{p}_{\mathbf{x}}^k = \mathbf{p}_{\mathbf{x}}^1 + \mathbf{p}_{\mathbf{x}}^{k-1}\mathbf{h_x}, \tag{10}$$

where $\mathbf{p}_{\mathbf{x}}^k$ denotes a $1 \times n_x$ vector of derivatives of p^k with respect to $\mathbf{x}$. This formula reproduces the first order expression derived in Hordahl et al. (2008).

Expression (9) also suggests that the easiest way to start the recursion is to approximate p^1 in the first step of the POP method. This gives the derivative $\left[p_{\mathbf{x}}^1\right]_{\alpha_1}$ required to compute the right hand side of (9). This procedure does not add extra state variables to the first perturbation step and will therefore be faster than the alternative of reporting the stochastic discount factor $\mathcal{M}$ mentioned above. Moreover, if the R-transformed level of the one period interest rate ytm is already given in the first perturbation step, then $\left[p_{\mathbf{x}}^1\right]_{\alpha_1}$ can be computed by differentiating $p^{t,1} = R^{-1}(1/R(ytm))$. For instance, using a log-transformation it holds that $p_{\mathbf{x}}^1 = -ytm_{\mathbf{x}}^1$.

The first order derivatives of bond prices with respect to σ are found in a similar way.[10] That is, we

[10] We know from Schmitt-Grohé & Uribe (2004) that these derivatives are zero. Nevertheless, we solve for these terms to make subsequent derivations of higher order derivatives more transparent.

exploit the fact that the derivative of $F^k(\mathbf{x},\sigma)$ with respect to σ evaluated at the deterministic steady state equals zero, i.e.

$$F_\sigma^k(\mathbf{x}_{ss},0) = E_t\Big\{R_p\left(p^k\right)\left[p_\sigma^k\right] - [\mathcal{M}_\sigma]\,R\left(p^{k-1}\right) \\ - \mathcal{M}R_p\left(p^{k-1}\right)\left(\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left([\mathbf{h}_\sigma]^{\gamma_1} + \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) + \left[p_\sigma^{k-1}\right]\right)\Big\} = 0. \quad (11)$$

For the one period bond, this reduces to

$$E_t\Big\{R_p\left(p^1\right)\left[p_\sigma^1\right]\Big\} = E_t\mathcal{M}_\sigma \quad (12)$$

as $R\left(p^0\right) = 1$, $E_t\left[\boldsymbol{\epsilon}_{t+1}\right] = 0$, and all the derivatives of p^0 are zero. The fact that $E_t\mathcal{M}_\sigma = 0$ implies $\left[p_\sigma^1\right] = 0$. Moreover, $\mathbf{h}_{\boldsymbol{\sigma}} = 0$ and this suffices to show that $p_\sigma^k = 0$ for $k = 2,3,...,K$.

2.3 Second order terms

This section shows how to compute all second order terms for bond prices. The procedure is similar to that used to compute all first order derivatives of bond prices. In particular, we use terms computed in the previous section, output from the first step of POP, and second order derivatives of $F^k(\mathbf{x},\sigma)$ evaluated in the deterministic steady state.

Starting with second order derivatives with respect to the state vector, we obtain

$$\begin{aligned}\left[F_{\mathbf{xx}}^k(\mathbf{x}_{ss},0)\right]_{\alpha_1,\alpha_2} = {} & R_{pp}\left(p^k\right)\left[p_{\mathbf{x}}^k\right]_{\alpha_2}\left[p_{\mathbf{x}}^k\right]_{\alpha_1} + R_p\left(p^k\right)\left[p_{\mathbf{xx}}^k\right]_{\alpha_1\alpha_2} \\ & - [\mathcal{M}_{\mathbf{xx}}]_{\alpha_1\alpha_2} R\left(p^{k-1}\right) - [\mathcal{M}_{\mathbf{x}}]_{\alpha_1} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2} \\ & - [\mathcal{M}_{\mathbf{x}}]_{\alpha_2} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1} \\ & -\mathcal{M}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1} \\ & -\mathcal{M}R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}[\mathbf{h}_{\mathbf{x}}]_{\alpha_2}^{\gamma_2}[\mathbf{h}_{\mathbf{x}}]_{\alpha_1}^{\gamma_1} \\ & -\mathcal{M}R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\mathbf{h}_{\mathbf{xx}}]_{\alpha_1\alpha_2}^{\gamma_1} = 0\end{aligned} \quad (13)$$

for $\alpha_1, \alpha_2, \gamma_1, \gamma_2 = 1, 2, ..., n_x$. To solve for second order bond price derivatives, we consider the case where the price of the one period bond is approximated in the first step of the POP method and focus on computing $p^k_{\mathbf{xx}}$ given $p^{k-1}_{\mathbf{xx}}$.[11] To evaluate the right hand side of equation (13) we need expressions for $\mathcal{M}, \mathcal{M}_{\mathbf{x}}$, and $\mathcal{M}_{\mathbf{xx}}$. The value of $\mathcal{M}$ equals $R(p^1)$ and $\mathcal{M}_{\mathbf{x}}$ is given by equation (7). The expression for $\mathcal{M}_{\mathbf{xx}}$ can be computed from equation (13) when $k = 1$

$$[\mathcal{M}_{\mathbf{xx}}]_{\alpha_1\alpha_2} = R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}, \tag{14}$$

as all derivatives of $p^0(\mathbf{x}, \sigma)$ are zero. Exploiting these findings in equation (13) gives

$$\begin{aligned} R_p\left(p^k\right)\left[p^k_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2} = & -R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1} \\ & + \left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right) R\left(p^{k-1}\right) \\ & + \left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \\ & + \left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\ & + R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\ & + R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\ & + R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2} \end{aligned} \tag{15}$$

for $k = 2, 3, ..., K$ and for $\alpha_1, \alpha_2, \gamma_1, \gamma_2 = 1, 2, ..., n_x$. For a log-transformation, the formula in (15) simplifies to

$$\mathbf{p}^k_{\mathbf{xx}} = \mathbf{p}^1_{\mathbf{xx}} + \mathbf{h}'_{\mathbf{x}}\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h}_{\mathbf{x}} + \textstyle\sum_{\gamma_1=1}^{n_x} p^{k-1}_{\mathbf{x}}(\gamma_1)\mathbf{h}_{\mathbf{xx}}(\gamma_1, :, :). \tag{16}$$

Here, we have adopted the notation used in Hordahl et al. (2008) to clearly demonstrate that equation (15) nests their second order expression. Using this notation, $A(\gamma_1, \gamma_2, \ldots, \gamma_N)$ denotes an element on the intersection of dimensions γ_1, γ_2, and γ_N in matrix $\mathbf{A}$ and colons refer to entire dimensions. For example, $\mathbf{h}_{\mathbf{xx}}(\gamma_1, :, :)$ is an $n_x \times n_x$ matrix of second order derivatives of the γ_1-th mapping of $\mathbf{h}$ evaluated at the steady state, and $\mathbf{p}^k_{\mathbf{xx}}$ is the $n_x \times n_x$ matrix of second order derivatives of p^k with

[11] Along the lines discussed in section 2.2 for $\mathcal{M}_{\mathbf{x}}$, the value of $\mathcal{M}_{\mathbf{xx}}$ can also be computed by second order differentiation of $\mathcal{M}$, or $\mathcal{M}_{\mathbf{xx}}$ may be reported directly in the first step of the POP method.

respect to $\mathbf{x}$.

To find $p^k_{\sigma\sigma}$, we differentiate $F^k(\mathbf{x},\sigma)$ twice with respect to σ and evaluate the expression in the deterministic steady state. Since this derivative is equal to zero, we get

$$\begin{aligned}[F_{\sigma\sigma}(\mathbf{x}_{ss},0)] \quad = \quad & E_t\Big\{-R_p\left(p^k\right)\left[p^k_{\sigma\sigma}\right]+[\mathcal{M}_{\sigma\sigma}]\,R\left(p^{k-1}\right) \\ & +[\mathcal{M}_{\sigma}]\,R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2} \\ & +[\mathcal{M}_{\sigma}]\,R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \\ & +\mathcal{M}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \\ & +\mathcal{M}R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\boldsymbol{\epsilon}_{t+1}]^{\phi_1} \\ & +\mathcal{M}R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}+\mathcal{M}R_{p}\left(p^{k-1}\right)\left[p^{k-1}_{\sigma\sigma}\right]\Big\}=0\end{aligned} \tag{17}$$

where $\gamma_1,\gamma_2 = 1,2,...,n_x$ and $\phi_1,\phi_2 = 1,2,...,n_\epsilon$. To simplify equation (17) we have relied on the fact that the terms $\mathbf{h}_\sigma$, $p^k_{\boldsymbol{\sigma}}$, and $p^k_{\mathbf{x}\boldsymbol{\sigma}}$ are known to be zero (Schmitt-Grohé & Uribe (2004)). Again, the important thing to observe is that equation (17) allows us to solve for $p^k_{\sigma\sigma}$. To show this, we first differentiate $\mathcal{M}$ with respect to σ to get

$$[\mathcal{M}_\sigma]=\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_2}[\mathbf{g}_{\mathbf{x}}]^{\beta_2}_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\epsilon}_{t+1}]^{\phi_2}. \tag{18}$$

To find an expression for $E_t[\mathcal{M}_{\sigma\sigma}]$ we consider the case in which $p^1_{\sigma\sigma}$ is reported in the first step of the POP method. Evaluating equation (17) at $k=1$ and exploiting the fact that all derivatives of $p^0(\mathbf{x},\sigma)$ are zero gives

$$E_t[\mathcal{M}_{\sigma\sigma}]=\left[p^1_{\sigma\sigma}\right]R_p\left(p^1\right). \tag{19}$$

Combining the results in (18) and (19) to evaluate (17) we get

$$\begin{aligned} R_p\left(p^k\right)\left[p_{\sigma\sigma}^k\right] &= \left[p_{\sigma\sigma}^1\right] R_p\left(p^1\right) R\left(p^{k-1}\right) \\ &+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2} \\ &+2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2} \\ &+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1} \\ &+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1} \\ &+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1} \\ &+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p_{\sigma\sigma}^{k-1}\right] \end{aligned} \tag{20}$$

As discussed previously, the derivatives of the stochastic discount factor $\mathcal{M}_{\mathbf{y}_{t+1}}$ and $\mathcal{M}_{\mathbf{x}_{t+1}}$ are straightforward to compute from the known functional form of $\mathcal{M}$. Applying a log-transformation makes it possible to simplify the above formula to

$$p_{\sigma\sigma}^k = p_{\sigma\sigma}^1 + p_{\sigma\sigma}^{k-1} + \mathbf{p}_{\mathbf{x}}^{k-1}\mathbf{h}_{\sigma\sigma} + \text{trace}\left(\boldsymbol{\eta}'\mathbf{p}_{\mathbf{xx}}^{k-1}\boldsymbol{\eta}\right) + \mathbf{p}_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'\left(\mathbf{p}_{\mathbf{x}}^{k-1}\right)' + 2\left(\boldsymbol{\lambda}_{\mathbf{x}} - \boldsymbol{\pi}_{\mathbf{x}}\right)\boldsymbol{\eta}\boldsymbol{\eta}'\left(\mathbf{p}_{\mathbf{x}}^{k-1}\right)', \tag{21}$$

when $\mathcal{M} = \beta\lambda_{t+1}/(\lambda_t\pi_{t+1})$. Here, β is the discount factor, λ_t denotes the marginal utility of consumption, and π_t stands for the inflation rate. We use $\boldsymbol{\lambda}_{\mathbf{x}}$ and $\boldsymbol{\pi}_{\mathbf{x}}$ to denote $1 \times n_x$ matrices of first order derivatives for λ_t and π_t with respect to $\mathbf{x}$ in the steady state, respectively. In this special case, formula (20) reproduces the second order expression derived in Hordahl et al. (2008).

2.4 Higher order approximations

The method described in the previous two subsections naturally generalizes to perturbation approximations of order higher than two. Third order terms are of significant economic interest because they allow for time-varying risk premia. We therefore provide explicit formulas for $\mathbf{p}_{\mathbf{xxx}}^k, \mathbf{p}_{\mathbf{x}\sigma\sigma}^k$, and $\mathbf{p}_{\boldsymbol{\sigma\sigma\sigma}}^k$, with the proof of $\mathbf{p}_{\mathbf{xx}\boldsymbol{\sigma}}^k = \mathbf{0}$ provided in Andreasen (2010*b*). In the interest of space, we only report simpler expressions for the log-transformation case in the body of the text and refer to the appendix for the general solutions corresponding to arbitrary $R(\cdot)$. As derived in the appendix

$$\begin{aligned} p^{k}_{\mathbf{xxx}}\left(\alpha_1,\alpha_2,\alpha_3\right) &= p^{1}_{\mathbf{xxx}}\left(\alpha_1,\alpha_2,\alpha_3\right) \\ &+\textstyle\sum_{\gamma_1=1}^{n_x}\mathbf{h_x}\left(\gamma_1,\alpha_1\right)\mathbf{h_x}\left(:,\alpha_2\right)'\mathbf{p}^{k-1}_{\mathbf{xxx}}\left(\gamma_1,:,:\right)\mathbf{h_x}\left(:,\alpha_3\right) \\ &+\mathbf{h_x}\left(:,\alpha_1\right)'\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h_{xx}}\left(:,\alpha_2,\alpha_3\right) \\ &+\mathbf{h_{xx}}\left(:,\alpha_1,\alpha_3\right)'\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h_x}\left(:,\alpha_2\right) \\ &+\mathbf{h_{xx}}\left(:,\alpha_1,\alpha_2\right)'\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h_x}\left(:,\alpha_3\right) \\ &+\mathbf{p}^{k-1}_{\mathbf{x}}\mathbf{h_{xxx}}\left(:,\alpha_1,\alpha_2,\alpha_3\right) \end{aligned} \tag{22}$$

for $k=2,3,...,K$ and $\alpha_1,\alpha_2,\alpha_3=1,2,\ldots,n_x$. The notation follows that in equation (16). When $\mathcal{M}=\beta\lambda_{t+1}/(\lambda_t\pi_{t+1})$, the general formulas for $\mathbf{p}^{k}_{\sigma\sigma\mathbf{x}}$ and $\mathbf{p}^{k}_{\sigma\sigma\sigma}$ reported in the appendix simplify to

$$\begin{aligned} \mathbf{p}^{k}_{\sigma\sigma\mathbf{x}} &= \mathbf{p}^{1}_{\sigma\sigma\mathbf{x}}-2\left(\boldsymbol{\lambda}_{\mathbf{x}}-\boldsymbol{\pi}_{\mathbf{x}}\right)\boldsymbol{\eta}\boldsymbol{\eta}'\left(\mathbf{p}^{k-1}_{\mathbf{x}}\right)'\mathbf{p}^{1}_{\mathbf{x}} \\ &+2\mathbf{p}^{k-1}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\eta}'\left(\boldsymbol{\lambda}'_{\mathbf{x}}\boldsymbol{\lambda}_{\mathbf{x}}-\boldsymbol{\lambda}'_{\mathbf{x}}\boldsymbol{\pi}_{\mathbf{x}}-\boldsymbol{\pi}'_{\mathbf{x}}\boldsymbol{\lambda}_{\mathbf{x}}+\boldsymbol{\pi}'_{\mathbf{x}}\boldsymbol{\pi}_{\mathbf{x}}+\mathbf{g}^{\lambda}_{\mathbf{xx}}-\mathbf{g}^{\pi}_{\mathbf{xx}}+\mathbf{p}^{k-1}_{\mathbf{xx}}\right)\mathbf{h_x} \\ &+2\mathbf{p}^{k-1}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\eta}'\left(\boldsymbol{\lambda}'_{\mathbf{x}}\boldsymbol{\pi}_{\mathbf{x}}-\boldsymbol{\lambda}'_{\mathbf{x}}\boldsymbol{\lambda}_{\mathbf{x}}\right) \\ &+2\left(\boldsymbol{\lambda}_{\mathbf{x}}-\boldsymbol{\pi}_{\mathbf{x}}\right)\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h_x} \\ &+\textstyle\sum_{\gamma_1=1}^{n_x}\boldsymbol{\eta}\left(\gamma_1,:\right)\boldsymbol{\eta}'\mathbf{p}^{k-1}_{\mathbf{xxx}}\left(\gamma_1,:,:\right)\mathbf{h_x} \\ &+\mathbf{h}'_{\sigma\sigma}\mathbf{p}^{k-1}_{\mathbf{xx}}\mathbf{h_x}+\mathbf{p}^{k-1}_{\mathbf{x}}\mathbf{h}_{\sigma\sigma\mathbf{x}}+\mathbf{p}^{k-1}_{\sigma\sigma\mathbf{x}}\mathbf{h_x} \end{aligned} \tag{23}$$

and

$$
\begin{aligned}
p_{\sigma\sigma\sigma}^{k} \;=\;& p_{\sigma\sigma\sigma}^{1}+\mathbf{p}_{\mathbf{x}}^{k-1}\mathbf{h}_{\sigma\sigma\sigma}+p_{\sigma\sigma\sigma}^{k-1} \qquad (24)\\
&+\sum_{\phi_1=1}^{n_e}3\left(\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right)\right)'\left(\boldsymbol{\lambda}_{\mathbf{x}}'\boldsymbol{\lambda}_{\mathbf{x}}-\boldsymbol{\lambda}_{\mathbf{x}}'\boldsymbol{\pi}_{\mathbf{x}}-\boldsymbol{\pi}_{\mathbf{x}}'\boldsymbol{\lambda}_{\mathbf{x}}+\boldsymbol{\pi}_{\mathbf{x}}'\boldsymbol{\pi}_{\mathbf{x}}\right)\boldsymbol{\eta}\left(:,\phi_1\right)\mathbf{p}_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\left(:,\phi_1\right)\\
&+\sum_{\phi_1=1}^{n_e}3\left(\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right)\right)'\left(\boldsymbol{\lambda}_{\mathbf{xx}}-\boldsymbol{\pi}_{\mathbf{xx}}\right)\boldsymbol{\eta}\left(:.\phi_1\right)\mathbf{p}_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\left(:,\phi_1\right)\\
&+\sum_{\phi_1=1}^{n_e}3\left(\boldsymbol{\lambda}_{\mathbf{x}}-\boldsymbol{\pi}_{\mathbf{x}}\right)\mathbf{g}_{\mathbf{x}}\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right)\mathbf{p}_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\left(:,\phi_1\right)\mathbf{p}_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\left(:,\phi_1\right)\\
&+\sum_{\phi_1=1}^{n_e}3\left(\boldsymbol{\lambda}_{\mathbf{x}}-\boldsymbol{\pi}_{\mathbf{x}}\right)\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right)\boldsymbol{\eta}\left(:,\phi_2\right)'\mathbf{p}_{\mathbf{xx}}^{k-1}\boldsymbol{\eta}\left(:,\phi_3\right)\\
&+\sum_{\phi_1=1}^{n_e}\mathbf{p}_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\left(:,\phi_1\right)\mathbf{p}_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\left(:,\phi_1\right)\mathbf{p}_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\left(:,\boldsymbol{\phi}_{\mathbf{1}}\right)m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right)\\
&+\sum_{\phi_1=1}^{n_e}3\mathbf{p}_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\left(:,\phi_1\right)\left(\boldsymbol{\eta}\left(:,\phi_1\right)m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right)\right)'\mathbf{p}_{\mathbf{xx}}^{k-1}\boldsymbol{\eta}\left(:,\phi_1\right)\\
&+\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}\left(:,\phi_1\right)'\mathbf{p}_{\mathbf{xxx}}^{k-1}\left(\gamma_1,:,:\right)\boldsymbol{\eta}\left(:,\phi_1\right)\boldsymbol{\eta}\left(\gamma_1,\phi_1\right)m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right)
\end{aligned}
$$

for $k = 2, 3, ..., K$. Here, $m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right)$ denotes the third moment of $\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)$ for $\phi_1 = 1, 2, ..., n_\epsilon$. Notably, formulas (22) - (24) extend the results in Hordahl et al. (2008) to third order perturbation approximations.

All the formulas derived in this paper are implemented in MATLAB and the codes are publicly available to facilitate their use.

2.5 Extensions

The setup considered above assumes that the model can be split into two distinct parts: one containing all equations in which bond prices beyond one period maturity do *not* appear and another consisting entirely of Euler equations for the remaining bond prices. However, the POP method may still be useful if this condition does not hold. To see how, consider the case in which one is interested in the dynamics of the 10 year yield curve but it is only possible to separate out bond price Euler equations of maturity greater than 5 years. To then apply our method, the model including bond prices of maturity up to 20 quarters (5 years) needs to be solved in the first step of POP. This gives all derivatives of bond prices for $k \leq 20$. The remaining derivatives, for bond prices with maturities between 5 and 10 years, i.e. $k \in \{21, 22, \ldots 40\}$, can then be computed in the second step by starting the recursions derived in this paper at $k = 20$.

Andreasen (2010*a*) presents another extension in which expected future short rates are computed in the second perturbation step, making it possible to efficiently solve for all risk premia. We also illustrate in the appendix that the POP method can be extended to the case in which no restrictions are imposed on the way shocks enter, as considered in *Dynare++* or *Perturbation AIM* (Kamenik (2005) and Swanson et al. (2005)). Furthermore, the scalar example presented in the appendix is derived using *Mathematica* and the underlying codes may be used to derive bond price approximations of orders higher than three.

3 Evaluating the computational gain

This section assesses the speed of the POP method and compares it to that of standard one-step perturbation. Clearly, both the absolute and relative performance of POP will depend on a number of factors. Those we focus on in this paper include the maximum maturity of bonds in the yield curve and the number of state variables in the model.

To illustrate the role played by the maximum bond maturity, we report results corresponding to nominal yield curves of maturities ranging from 5 to 20 years.[12] The relevance of the number of state variables is shown by reporting times corresponding to the DSGE model of Rudebusch & Swanson (2008) with 9 states as well as a version of the model by De Paoli et al. (2007) with 15 state variables. Both models are approximated to third order using *Dynare++*.[13]

The absolute times are reported in seconds, while the relative computational gain from POP is measured as

$$\text{Speed gain} = \frac{\text{Computing time using the one-step perturbation method}}{\text{Computing time using the POP method}}.$$

[12] The case of the 20 year yield curve is seldom considered in the literature, with the 10 year yield curve being the benchmark. However, from a computational perspective, approximating the 20 year yield curve is equivalent to: i) computing jointly the 10 year nominal and real yield curves, or ii) computing the 10 year yield curve and the corresponding term premia.

[13] Our implementation of the POP method does not exploit multi-threading, and we therefore do not use it in *Dynare++*. Note however, that by using MATLAB to code our routines we are already sacrificing performance relative to the more efficient *C++* implementation of *Dynare++*.

Table 1 reports Monte Carlo estimates of the time and the speed gain based on 20 replications for a third order approximation. The POP method turns out to be 23 times faster than the one-step perturbation method for the model by Rudebusch & Swanson (2008) with a 10 year yield curve. This number increases to 139 with a 20 year yield curve. The computational gains from the POP method in the model by De Paoli et al. (2007) are somewhat lower, with the corresponding figures equal to 14 and 61, respectively. As shown in Andreasen (2010*a*), the speed gains involved are sufficient to make estimation of medium-scale DSGE models with a whole yield curve approximated to third order feasible.

< Table 1 about here >

4 Comparing solution accuracy

Our proposed POP method is faster to execute than traditional one-step perturbation, but there are other approximation methods which have become popular, in part due to their speed. This section compares the accuracy of the POP method to that of three well-known alternatives. In doing so, we add to the results of Arouba et al. (2005) and Caldara et al. (2009) by examining the accuracy of different interest rates approximations.

The first alternative considered is the first order log-normal method proposed by Jermann (1998). Its accuracy is potentially compromised by the fact that it only includes some second order terms in bond price approximations. The next alternative is Doh's (2007) second order log-normal method. This extends Jermann's approach by combining second order perturbation approximations with bond prices derived from the log-normal assumption. This approach is subject to similar type of criticisms as Jermann's method, because it only includes some third and fourth order terms (see Andreasen (2009)). The final alternative considered is the 'consol' method proposed in Rudebusch & Swanson (2008) where consol bonds are used to compute yields.[14] Its accuracy may be adversely affected by the fact that consol bonds and zero-coupon bonds have very different cash flows, and matching the

[14] As shown by Rudebusch & Swanson (2008), prices of consols satisfy simple recursive relationships which make them fast to approximate.

first order concept of duration potentially allows for different higher order properties of these bond prices.

To assess the accuracy of the aforementioned methods, we use closed-form solutions for zero-coupon bond prices in a consumption endowment model with external habits derived in Zabczyk (2010). The habit specification nests that considered in Burnside (1998) and Tsionas (2003) and has become a standard ingredient of many models. For example, the papers of De Paoli et al. (2007) and Hordahl et al. (2008) amongst others, use habits when studying the properties of bond prices in DSGE models.

We proceed by briefly introducing the habit model in section 4.1. Section 4.2 then compares the approximation accuracy of the POP method to that of the three alternatives mentioned above.

4.1 The consumption endowment model with habits

We consider a representative agent with the standard utility function

$$U_0 = \sum_{t=0}^{\infty} \beta^t E_0 \left[\frac{(C_t - hC_{t-1})^{1-\gamma} - 1}{1-\gamma} \right],$$

where C_t is consumption and $h \in [0,1]$ controls the degree of external habit formation. Consumption growth is defined as $x_t := \ln(C_t/C_{t-1})$, and x_t is assumed to follow an AR(1) process, i.e.

$$x_t = (1-\rho)\,\mu + \rho x_{t-1} + \xi_t \tag{25}$$

where $\xi_t \sim \mathcal{NID}\left(0, \sigma_\xi^2\right)$. This implies the following expression for the stochastic discount factor

$$\mathcal{M}_{t+1} \equiv \beta \frac{(C_{t+1} - hC_t)^{-\gamma}}{(C_t - hC_{t-1})^{-\gamma}} = \left(\frac{1 - h\exp\{-x_{t+1}\}}{1 - h\exp\{-x_t\}} \right)^{-\gamma} \exp\{-\gamma x_{t+1}\},$$

and a closed-form solution for zero-coupon bond prices of the form (Zabczyk (2010))

$$P_t^k = \big(1 - h\exp(-x_t)\big)^{\gamma}\beta^k \exp\Big\{-\gamma\big(k\mu + (x_t-\mu)\frac{\rho(1-\rho^k)}{(1-\rho)}\big)\Big\}$$
$$\times \sum_{n=0}^{+\infty}\left(\begin{array}{c}-\gamma\\ n\end{array}\right)(-h)^n \exp\Big\{-n\mu - n(x_t-\mu)\rho^k\Big\}\prod_{j=0}^{k}\mathcal{L}_\xi\Big(-\gamma\frac{(1-\rho^j)}{(1-\rho)} - n\frac{\rho^j - 0^j}{\rho}\Big).$$

Here, $\mathcal{L}_\xi$ is the Laplace transform of ξ, and $\left(\begin{array}{c}\alpha\\ n\end{array}\right)$ denotes a generalized binomial coefficient, i.e.

$$\left(\begin{array}{c}\alpha\\ n\end{array}\right) := \prod_{k=1}^{n}(\alpha-k+1)/k, \text{ for } n>0 \text{ and } \left(\begin{array}{c}\alpha\\ 0\end{array}\right) := 1,$$

where $\alpha \in \mathbb{R}$ and $n \in \mathcal{N}$. Approximations to bond prices and yields derived using the first and second order log-normal methods and consol bonds are presented in a technical appendix.

The model is calibrated as follows. We let $\beta = 0.9995$ and set $h = 0.7$ based on the findings in Christiano, Eichenbaum & Evans (2005) and Smets & Wouters (2007). The coefficients in the consumption process in (25) are determined from an OLS regression for US nondurable consumption in the period 1947Q1 to 2009Q2. This implies $\mu = 0.0062$, $\rho = 0.0633$, and $\sigma_\xi = 6.4379 \times 10^{-5}$. Two values are considered for the curvature parameter gamma. Our first choice is to let $\gamma = 1$ which corresponds to standard log-preferences. Our second choice is to set γ to 5 and serves to explore the effects of stronger non-linearities.

4.2 The accuracy of various approximation methods

Figure 1 plots the benchmark 10 year interest rate as a function of consumption growth when $\gamma = 1$. The solid red line represents the exact solution and the other lines correspond to various approximation methods. Approximated solutions from one-step perturbation and the POP method are identical and are referred to as ‘perturbation method’ throughout this section. Further, under our calibration, the functions corresponding to second order perturbations are indistinguishable from the second order log-normal ones, and we therefore only plot the former.

We first note that for the range of consumption growth values considered, the third order perturbation method delivers an approximation which is hard to distinguish from the exact solution. The second order perturbation method is also quite accurate, with small deviations from the exact solution only visible for consumption growth deviations exceeding ± 0.08. The consol method generates larger approximation errors with clearer deviations from the exact solution. Importantly, for $\gamma = 1$, all of these methods are more accurate than the first order log-normal method.

$<$ Figure 1 about here $>$

In Figure 2, we turn to the case of stronger non-linearities with $\gamma = 5$. Again, the third order perturbation approximation is almost indistinguishable from the exact solution. Figure 2 also shows that the second order perturbation approximation is fairly accurate. The consol method, on the other hand, does worse than the first order log-normal method. It is also interesting to note that moving from a second order to a third order approximation in the consol method does not improve its accuracy.

$<$ Figure 2 about here $>$

These observations are also confirmed by Table 2 which reports the root mean squared errors of the approximations in Figure 1 and 2. The table also shows that for our model and the chosen calibration, third order perturbation clearly outperforms all the alternative methods considered.

$<$ Table 2 about here $>$

To evaluate the impact of the approximation errors reported above, we now focus on the first four moments of the benchmark 10 year interest rate. These moments are compared in Tables 3 and 4 which correspond to $\gamma = 1$ and $\gamma = 5$, respectively. We see that both versions of the perturbation method and the log-normal method reproduce the correct annualized mean. However, the consol method overestimates it by approximately 20 basis points when $\gamma = 1$ and by 24 basis points when $\gamma = 5$. The consol method also underestimates the standard deviation by more than all the other methods. Values of skewness and kurtosis are closely matched by the two perturbation methods and the second order log-normal method, but not by the consol and first order log-normal formulas.

$<$ Table 3 about here $>$

< Table 4 about here >

Summarizing, third order perturbations – and hence the POP method – approximate the 10 year interest rate most accurately in the examples considered. The precision of the second order log-normal method is very similar to that of the second order perturbation method, and both outperform the first order log-normal approximation. We also find that the consol method gives a less accurate approximation, and we show that, even at third order, it may be less accurate than first order log-normal formula.

5 Conclusion

This paper proposes a new method of computing bond price approximations. The approach is applicable to a wide class of DSGE models and uses the perturbation principle sequentially. While the final formulas for bond prices *exactly* match those derived using the standard one-step perturbation method, a simulation study documents that execution times can be more than one hundred times shorter. In general, the exact improvement in speed depends on the maturity of the approximated yield curve and the number of state variables in the DSGE model.

The paper also assesses the accuracy of the POP/perturbation method implemented up to third order in a consumption endowment model with habits. Our results show that second and third order approximations to the 10 year interest rate are more accurate than those of popular alternatives and can be hard to distinguish from the true solution. It is also shown that interest rates approximated from prices of consol bonds can be less precise, even at third order, than those computed using the first order log-normal approach.

A Third order terms for bond prices

This appendix derives the third order terms for bond prices in the framework of Schmitt-Grohé & Uribe (2004).

A.1 Derivative of p^k with respect to $(\mathbf{x}, \mathbf{x}, \mathbf{x})$

Applying the chain rule to the definition of F^k one can show that $[F_{\mathbf{xxx}}(\mathbf{x}_{ss}, 0)]_{\alpha_1\alpha_2\alpha_3} = 0$ equals

$$
\begin{aligned}
R_p\left(p^k\right)\left[p^k_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3} &= -R_{ppp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1} \\
&-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_1} \\
&-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{xx}}\right]_{\alpha_1\alpha_3} \\
&-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\mathbf{xx}}\right]_{\alpha_1\alpha_2} \\
&+\left[\mathcal{M}_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3} R\left(p^{k-1}\right) \\
&+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3} \\
&+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_3} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right)/R\left(p^0\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3} \\
&+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_2,\alpha_3} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right)/R\left(p^0\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\mathbf{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3} \\
&+\left[p^1_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1} \\
&+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h}_{\mathbf{x}}\right]^{\gamma_1}_{\alpha_1}
\end{aligned}
$$

$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$

$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$

$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$

$+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$

$+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$

$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$

$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}$

Note that we can also eliminate $\left[\mathcal{M}_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}$ from this expression. Again, the trick is to observe that for $k=1$ we have $P^0=1$ for all values of $(\mathbf{x}_t,\sigma)$ and so all derivatives have to equal zero. Thus

$R_p\left(p^1\right)\left[p^1_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}=-R_{ppp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}$

$-R_{pp}\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}$

$-R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_3}$

$-R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}$

$+\left[\mathcal{M}_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}R\left(p^0\right)$

$\Updownarrow$

$(R_p\left(p^1\right)\left[p^1_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}+R_{ppp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}$

$R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_2})=\left[\mathcal{M}_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}$

because $R\left(p^0\right)=1$.

Thus we get for $k>1$

$R_p\left(p^k\right)\left[p^k_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}=-R_{ppp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1}$

$-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_2\alpha_3}\left[p^k_{\mathbf{x}}\right]_{\alpha_1}$

$-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_2}\left[p^k_{\mathbf{x}}\right]_{\alpha_1\alpha_3}$

$-R_{pp}\left(p^k\right)\left[p^k\right]_{\alpha_3}\left[p^k_{\mathbf{xx}}\right]_{\alpha_1\alpha_2}$

$+(R_p\left(p^1\right)\left[p^1_{\mathbf{xxx}}\right]_{\alpha_1\alpha_2\alpha_3}+R_{ppp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_2\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}$

$+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_2})R\left(p^{k-1}\right)$

$+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1,\alpha_2}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}$

$+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_1\alpha_3}+R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_1}\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}$

$+\left[p^1_{\mathbf{x}}\right]_{\alpha_1}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}$

$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_1} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}$$
$$+\left(R_p\left(p^1\right)\left[p^1_{\mathbf{xx}}\right]_{\alpha_2,\alpha_3} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\mathbf{x}}\right]_{\alpha_2}\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right)/R\left(p^0\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_2} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+R\left(p^1\right) R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_{xx}}\right]^{\gamma_2}_{\alpha_2\alpha_3}\left[\mathbf{h_x}\right]^{\gamma_1}_{\alpha_1}$$
$$+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}\left[\mathbf{h_x}\right]^{\gamma_2}_{\alpha_2}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_3}$$
$$+\left[p^1_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\mathbf{h_{xx}}\right]^{\gamma_1}_{\alpha_1\alpha_2}$$
$$+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h_{xxx}}\right]^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}$$

With a log-transformation $R\left(p^{t,k}\right) = M^k$, $R_p\left(p^{t,k}\right) = M^k$, $R_{pp}\left(p^{t,k}\right) = M^k$ and $R_{ppp}\left(p^{t,k}\right) = M^k$ in the deterministic steady state. Using the expressions for first and second order derivatives of bond prices derived above, we get, after simplifying, the expression stated in the body of the text.

A.2 Derivative of p^k with respect to $(\sigma, \sigma, \mathbf{x})$

It is possible to show that $\left[F_{\sigma\sigma\mathbf{x}}\left(\mathbf{x}_{ss}, 0\right)\right]_{\alpha_3} = 0$ implies

$$E_t\{-R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\sigma\sigma}\right] - R_p\left(p^k\right)\left[p^k_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}$$

$$
\begin{aligned}
&+\left[\mathcal{M}_{\sigma\sigma\mathbf{x}}\right] R\left(p^{k-1}\right)+\left[\mathcal{M}_{\sigma\sigma}\right] R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3} \\
&+2\left[\mathcal{M}_{\sigma\mathbf{x}}\right]_{\alpha_3} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2} \\
&+2\left[\mathcal{M}_{\sigma}\right] R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2} \\
&+2\left[\mathcal{M}_{\sigma}\right] R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2} \\
&+\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_3} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\
&+\mathcal{M} R_{ppp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\
&+\mathcal{M} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\
&+\mathcal{M} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\
&+\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\
&+\mathcal{M} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\
&+\mathcal{M} R_p\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1} \\
&+\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1} \\
&+\mathcal{M} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1} \\
&+\mathcal{M} R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1} \\
&+\mathcal{M} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}^{\gamma_1} \\
&+\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_3} R_p\left(p^{k-1}\right)\left[p_{\sigma\sigma}^{k-1}\right] \\
&+\mathcal{M} R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\sigma\sigma}^{k-1}\right] \\
&+\mathcal{M} R_p\left(p^{k-1}\right)\left[p_{\sigma\sigma\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\}=0
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
&R_p\left(p^k\right)\left[p_{\sigma\sigma\mathbf{x}}^k\right]_{\alpha_3}=-R_{pp}\left(p^k\right)\left[p_{\mathbf{x}}^k\right]_{\alpha_3}\left[p_{\sigma\sigma}^k\right] \\
&\quad+E_t\left[\mathcal{M}_{\sigma\sigma\mathbf{x}}\right] R\left(p^{k-1}\right) \\
&\quad+\left[p_{\sigma\sigma}^1\right] R_p\left(p^1\right)/R\left(p^0\right) R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3} \\
&\quad+2E_t\left(\left[\mathcal{M}_{\sigma\mathbf{x}}\right]_{\alpha_3} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \\
&\quad+2E_t\left(\left[\mathcal{M}_{\sigma}\right] R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \\
&\quad+2E_t\left(\left[\mathcal{M}_{\sigma}\right] R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right) \\
&\quad+\left[p_{\mathbf{x}}^1\right]_{\alpha_3} R_p\left(p^1\right)/R\left(p^0\right) R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}[\mathbf{I}]_{\phi_2}^{\phi_1}
\end{aligned}
$$

$$+R\left(p^{1}\right)R_{ppp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}$$

$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}$$

$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}$$

$$+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_{p}\left(p^{1}\right)/R\left(p^{0}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}$$

$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}$$

$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\mathbf{I}\right]_{\phi_2}^{\phi_1}$$

$$+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_{p}\left(p^{1}\right)/R\left(p^{0}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}$$

$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}$$

$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}$$

$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}^{\gamma_1}$$

$$+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_{p}\left(p^{1}\right)/R\left(p^{0}\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma}^{k-1}\right]$$

$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\sigma\sigma}^{k-1}\right]$$

$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}$$

where we have used

$\mathcal{M}=R\left(p^{1}\right)$

$\left[\mathcal{M}_{\mathbf{x}}\right]_{\alpha_3}=\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_{p}\left(p^{1}\right)/R\left(p^{0}\right)$

$E_t\left[\mathcal{M}_{\sigma\sigma}\right]=\left[p_{\sigma\sigma}^{1}\right]R_{p}\left(p^{1}\right)/R\left(p^{0}\right)$

We now compute the terms with derivatives of σ. Here we recall that

$\left[\mathcal{M}_{\sigma}\right]\equiv(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1})$

So

$$2E_t\left(\left[\mathcal{M}_{\sigma}\right]R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$

$$=2E_t\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$

$$+2E_t\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$

$$=2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g}_{\mathbf{x}}\right]_{\gamma_1}^{\beta_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2}$$

$$+2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]_{\phi_1}^{\gamma_1}R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]_{\phi_2}^{\gamma_2}\left[\mathbf{I}\right]_{\phi_1}^{\phi_2}$$

and

$$2E_t\left(\left[\mathcal{M}_\sigma\right]R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$
$$=2E_t\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$
$$+2E_t\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$

$$=2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2E_t\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$

To compute the $\left[\mathcal{M}_{\sigma\mathbf{x}}\right]_{\alpha_3}$ term, we need to find an expression for $\left[\mathcal{M}_{\sigma\mathbf{x}}\right]_{\alpha_3}$. Hence

$$\left[\mathcal{M}_{\sigma\mathbf{x}}\right]_{\alpha_3}=\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$

$$+\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)$$
$$\times\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$

So

$$2E_t\left(\left[\mathcal{M}_{\sigma\mathbf{x}}\right]_{\alpha_3}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\right)$$
$$=2E_t\{\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)$$
$$\times\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\}$$

$$=2\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\right)$$
$$\times\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$
$$+2\left(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\mathbf{h_x}\right]^{\gamma_3}_{\alpha_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\right)$$
$$\times\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{I}\right]^{\phi_2}_{\phi_1}$$

We finally note that $E_t\left([\mathcal{M}_{\sigma\sigma\mathbf{x}}]_{\alpha_3}\right)$ can be solved for and then substituted out by exploiting the fact that for $k=1$ we have $P^0=1$ for all values of $(\mathbf{x}_t,\sigma)$ and so all derivatives have to equal zero. Thus

$$R_p\left(p^1\right)\left[p^1_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3} = -R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\sigma\sigma}\right] + E_t\left([\mathcal{M}_{\sigma\sigma\mathbf{x}}]_{\alpha_3}\right)R\left(p^0\right)$$

$$\Updownarrow$$

$$\Big(R_p\left(p^1\right)\left[p^1_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\sigma\sigma}\right]\Big)/R\left(p^0\right) = E_t\left([\mathcal{M}_{\sigma\sigma\mathbf{x}}]_{\alpha_3}\right)$$

So for $k>1$ we get

$$\begin{aligned}
&R_p\left(p^k\right)\left[p^k_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3} = -R_{pp}\left(p^k\right)\left[p^k_{\mathbf{x}}\right]_{\alpha_3}\left[p^k_{\sigma\sigma}\right] \\
&\quad+\Big(R_p\left(p^1\right)\left[p^1_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3} + R_{pp}\left(p^1\right)\left[p^1_{\mathbf{x}}\right]_{\alpha_3}\left[p^1_{\sigma\sigma}\right]\Big)R\left(p^{k-1}\right)/R\left(p^0\right) \\
&\quad+\left[p^1_{\sigma\sigma}\right]R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} \\
&\quad+2\Big(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_t}\right]_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_t}\right]_{\beta_1\alpha_3}\Big) \\
&\quad\times[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{I}]^{\phi_2}_{\phi_1} \\
&\quad+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g}_{\mathbf{xx}}]^{\beta_1}_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{I}]^{\phi_2}_{\phi_1} \\
&\quad+2\Big(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_t}\right]_{\gamma_1\beta_3}[\mathbf{g}_{\mathbf{x}}]^{\beta_3}_{\alpha_3} + \left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3} + \left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_t}\right]_{\gamma_1\alpha_3}\Big) \\
&\quad\times[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{I}]^{\phi_2}_{\phi_1} \\
&\quad+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{I}]^{\phi_2}_{\phi_1} \\
&\quad+2\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{I}]^{\phi_2}_{\phi_1} \\
&\quad+2\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g}_{\mathbf{x}}]^{\beta_1}_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{I}]^{\phi_2}_{\phi_1} \\
&\quad+2E_t\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\mathbf{I}]^{\phi_2}_{\phi_1} \\
&\quad+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2} \\
&\quad+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2} \\
&\quad+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2} \\
&\quad+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2} \\
&\quad+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2} \\
&\quad+R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_2}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2} \\
&\quad+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}[\mathbf{h}_{\mathbf{x}}]^{\gamma_3}_{\alpha_3}[\boldsymbol{\eta}]^{\gamma_2}_{\phi_2}[\boldsymbol{\eta}]^{\gamma_1}_{\phi_1}[\mathbf{I}]^{\phi_1}_{\phi_2} \\
&\quad+\left[p^1_{\mathbf{x}}\right]_{\alpha_3}R_p\left(p^1\right)/R\left(p^0\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}
\end{aligned}$$

$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}$$
$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[\mathbf{h}_{\sigma\sigma}\right]^{\gamma_1}$$
$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\mathbf{x}}\right]_{\alpha_3}^{\gamma_1}$$
$$+\left[p_{\mathbf{x}}^{1}\right]_{\alpha_3}R_{p}\left(p^{1}\right)/R\left(p^{0}\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma}^{k-1}\right]$$
$$+R\left(p^{1}\right)R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}\left[p_{\sigma\sigma}^{k-1}\right]$$
$$+R\left(p^{1}\right)R_{p}\left(p^{k-1}\right)\left[p_{\sigma\sigma\mathbf{x}}^{k-1}\right]_{\gamma_3}\left[\mathbf{h}_{\mathbf{x}}\right]_{\alpha_3}^{\gamma_3}$$

For a logarithm transformation $R\left(p^{t,k}\right)=M^{k}$, $R_{p}\left(p^{t,k}\right)=M^{k}$, $R_{pp}\left(p^{t,k}\right)=M^{k}$, and $R_{ppp}\left(p^{t,k}\right)=M^{k}$. Using the expressions for first and second order derivatives of bond prices derived above, we get, after simplifying,

$$p_{\sigma\sigma\mathbf{x}}^{k}\left(1,\alpha_3\right)=$$
$$-2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{y}_{t+1}}\mathbf{g}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\eta}'\left(p_{\mathbf{x}}^{k-1}\right)'p_{\mathbf{x}}^{1}\left(1,\alpha_3\right)$$
$$-2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{x}_{t+1}}\boldsymbol{\eta}\boldsymbol{\eta}'\left(p_{\mathbf{x}}^{k-1}\right)'p_{\mathbf{x}:}^{1}\left(1,\alpha_3\right)$$
$$+p_{\sigma\sigma\mathbf{x}}^{1}\left(1,\alpha_3\right)$$
$$+2\mathcal{M}^{-1}p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'\left(\mathbf{g}_{\mathbf{x}}\right)'$$
$$\times\left(\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\mathbf{g}_{\mathbf{x}}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t}}\mathbf{g}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t}}\left(:,\alpha_3\right)\right)$$
$$+\sum_{\beta_1=1}^{n_y}2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{y}_{t+1}}\left(1,\beta_1\right)p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{g}_{\mathbf{xx}}\left(\beta_1,:,:\right)\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+2\mathcal{M}^{-1}p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'$$
$$\times\left(\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\mathbf{g}_{\mathbf{x}}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t}}\mathbf{g}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)+\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t}}\left(:,\alpha_3\right)\right)$$
$$+2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{y}_{t+1}}\mathbf{g}_{\mathbf{x}}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+2\mathcal{M}^{-1}\mathcal{M}_{\mathbf{x}_{t+1}}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+p_{\mathbf{x}}^{k-1}\boldsymbol{\eta}\boldsymbol{\eta}'p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+\sum_{\gamma_1=1}^{n_x}\boldsymbol{\eta}\left(\gamma_1,:\right)\boldsymbol{\eta}'p_{\mathbf{xxx}}^{k-1}\left(\gamma_1,:,:\right)\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$
$$+\left(\mathbf{h}_{\sigma\sigma}\right)'p_{\mathbf{xx}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_k\right)$$
$$+p_{\mathbf{x}}^{k-1}\mathbf{h}_{\sigma\sigma\mathbf{x}}\left(:,\alpha_3\right)$$
$$+p_{\sigma\sigma\mathbf{x}}^{k-1}\mathbf{h}_{\mathbf{x}}\left(:,\alpha_3\right)$$

A.3 Derivative of p^{k} with respect to (σ,σ,σ)

It is possible to show that $F_{\sigma\sigma\sigma}\left(\mathbf{x}_{ss},0\right)=0$ implies

$\left[F_{\sigma\sigma\sigma}\left(\mathbf{x}_{ss},0\right)\right]=E_t\{-R_p\left(p^{k}\right)\left[p_{\sigma\sigma\sigma}^{k}\right]$

$$+\left[\mathcal{M}_{\sigma\sigma\sigma}\right] R\left(p^{k-1}\right)$$
$$+3\left[\mathcal{M}_{\sigma\sigma}\right] R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+3\left[\mathcal{M}_{\sigma}\right] R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+3\left[\mathcal{M}_{\sigma}\right] R_p\left(p^{k-1}\right)\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_2\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}$$
$$+R\left(p^1\right) R_{ppp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+3R\left(p^1\right) R_{pp}\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}\left[p_{\mathbf{xx}}^{k-1}\right]_{\gamma_1\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p_{\mathbf{xxx}}^{k-1}\right]_{\gamma_1\gamma_2\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_2}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}$$
$$+R\left(p^1\right) R_p\left(p^{k-1}\right)\left[p_{\sigma\sigma\sigma}^{k-1}\right]\}=0$$

We next use the expression for $[\mathcal{M}_\sigma]$ found previously. We also have from differentiation of $\mathcal{M}$ that

$$[\mathcal{M}_{\sigma\sigma}]=(\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]_{\gamma_3}^{\beta_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}+\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3})[\mathbf{g}_{\mathbf{x}}]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g}_{\mathbf{xx}}]_{\gamma_1\gamma_3}^{\beta_1}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g}_{\mathbf{x}}]_{\gamma_1}^{\beta_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$
$$+\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}[\mathbf{g}_{\sigma\sigma}]^{\beta_1}$$
$$+\left[\mathcal{M}_{\mathbf{y_t}}\right]_{\beta_1}[\mathbf{g}_{\sigma\sigma}]^{\beta_1}$$
$$+(\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{y}_{t+1}}\right]_{\gamma_1\beta_3}[\mathbf{g}_{\mathbf{x}}]_{\gamma_3}^{\beta_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3}+\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_3})[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1}\left[\boldsymbol{\epsilon}_{t+1}\right]^{\phi_1}$$
$$+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}[\mathbf{h}_{\sigma\sigma}]^{\gamma_1}$$

For $[\mathcal{M}_{\sigma\sigma\sigma}]$, we exploit the fact $P^0=1$ for all values of $(\mathbf{x}_t,\sigma)$ and so all derivatives have to equal zero. Thus $R_p\left(p^1\right)\left[p_{\sigma\sigma\sigma}^1\right]=E_t\left\{[\mathcal{M}_{\sigma\sigma\sigma}]\right\}.$

To evaluate the expectations in the term for $[F_{\sigma\sigma\sigma}(\mathbf{x}_{ss},0)]$, we define

$$\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}=\begin{cases} m^3\left(\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)\right) & \text{if } \phi_1=\phi_2=\phi_3 \\ 0 & \text{otherwise}\end{cases}$$

where $m^3\left(\boldsymbol{\epsilon}_{t+1}\right)$ denotes the third moment of $\boldsymbol{\epsilon}_{t+1}\left(\phi_1\right)$ for $\phi_1=1,2,...,n_\epsilon$. Notice that $\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)$ is an $n_\epsilon\times n_\epsilon\times n_\epsilon$ matrix. Following some simplifications we finally get

$$R_p\left(p^k\right)\left[p_{\sigma\sigma\sigma}^k\right]=$$
$$+R_p\left(p^1\right)\left[p_{\sigma\sigma\sigma}^1\right] R\left(p^{k-1}\right)$$
$$+3\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}[\mathbf{g}_{\mathbf{x}}]_{\gamma_3}^{\beta_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\mathbf{g}_{\mathbf{x}}]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$
$$+6\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}[\boldsymbol{\eta}]_{\phi_3}^{\gamma_3}[\mathbf{g}_{\mathbf{x}}]_{\gamma_1}^{\beta_1}[\boldsymbol{\eta}]_{\phi_1}^{\gamma_1} R_p\left(p^{k-1}\right)\left[p_{\mathbf{x}}^{k-1}\right]_{\gamma_2}[\boldsymbol{\eta}]_{\phi_2}^{\gamma_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]_{\phi_2\phi_3}^{\phi_1}$$

$$+3\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\right)\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}$$
$$+3\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\right)\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}$$
$$+R\left(p^1\right)R_{ppp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3R\left(p^1\right)R_{pp}\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+R\left(p^1\right)R_p\left(p^{k-1}\right)\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}$$
$$+R\left(p^1\right)R_p\left(p^{k-1}\right)p^{k-1}_{\sigma\sigma\sigma}$$

For a logarithm transformation, it is straightforward to show that

$$p^k_{\sigma\sigma\sigma}=p^1_{\sigma\sigma\sigma}+3\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{y}_{t+1}}\right]_{\beta_1\beta_3}\left[\mathbf{g_x}\right]^{\beta_3}_{\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+6\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{y}_{t+1}\mathbf{x}_{t+1}}\right]_{\beta_1\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_{xx}}\right]^{\beta_1}_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\mathcal{M}^{-1}\left[\mathcal{M}_{\mathbf{x}_{t+1}\mathbf{x}_{t+1}}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\mathcal{M}^{-1}\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\right)\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}$$
$$+3\mathcal{M}^{-1}\left(\left[\mathcal{M}_{\mathbf{y}_{t+1}}\right]_{\beta_1}\left[\mathbf{g_x}\right]^{\beta_1}_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}+\left[\mathcal{M}_{\mathbf{x}_{t+1}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\right)\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}$$
$$+\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+3\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_2}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[p^{k-1}_{\mathbf{xx}}\right]_{\gamma_1\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+\left[p^{k-1}_{\mathbf{xxx}}\right]_{\gamma_1\gamma_2\gamma_3}\left[\boldsymbol{\eta}\right]^{\gamma_3}_{\phi_3}\left[\boldsymbol{\eta}\right]^{\gamma_2}_{\phi_2}\left[\boldsymbol{\eta}\right]^{\gamma_1}_{\phi_1}\left[\mathbf{m}^3\left(\boldsymbol{\epsilon}_{t+1}\right)\right]^{\phi_1}_{\phi_2\phi_3}$$
$$+\left[p^{k-1}_{\mathbf{x}}\right]_{\gamma_1}\left[\mathbf{h}_{\sigma\sigma\sigma}\right]^{\gamma_1}+p^{k-1}_{\sigma\sigma\sigma}$$

B The POP method when perturbing the state variables and the innovations

Consider the class of DSGE models with the following set of equilibrium and market clearing conditions:

$$E_t\left[f\left(z_{t+1},z_t,z_{t-1},\sigma u_t\right)\right]=0$$

where σ is the perturbation parameter. The vector z_t contains all the endogenous variables and u_t is the vector of disturbances with the property $u_t \sim \mathcal{IID}(0,\Sigma)$. The general solution is given by $z_t = g(z_{t-1}, u_t, \sigma)$.

When the approximation is done in levels, the fundamental pricing equation implies

$$P^n(z_{t-1}, u_t, \sigma) = E_t\left[\mathcal{M}(g_t(z_{t-1}, u_t, \sigma), u_{t+1}, \sigma) P^{n-1}(g_t(z_{t-1}, u_t, \sigma), u_{t+1}, \sigma)\right].$$

The recursive solution for a third order approximation of bond prices is then given by

$$P^n = \underbrace{\mathcal{M}p^{n-1}}_{P^n} + u_t \underbrace{g_u \mathcal{A}_1^{n-1}}_{P_u^n} + z_t \underbrace{g_z \mathcal{A}_1^{n-1}}_{P_z^n}$$

$$+\frac{1}{2}u_t^2 \underbrace{\left(g_{uu}\mathcal{A}_1^{n-1} + g_u^2 \mathcal{A}_2^{n-1}\right)}_{P_{uu}^n} + z_t u_t \underbrace{\left(g_{zu}\mathcal{A}_1^{n-1} + g_u g_z \mathcal{A}_2^{n-1}\right)}_{P_{zu}^n} + \frac{1}{2}z_t^2 \underbrace{\left(g_{zz}\mathcal{A}_1^{n-1} + g_z^2 \mathcal{A}_2^{n-1}\right)}_{P_{zz}^n}$$

$$+\frac{1}{2}\sigma^2 \underbrace{\left(p^{n-1}\left(\Sigma\mathcal{M}_{uu} + \mathcal{M}_{\sigma\sigma}\right) + 2\Sigma\mathcal{M}_u p_u^{n-1} + \mathcal{M}\left(\Sigma p_{uu}^{n-1} + p_{\sigma\sigma}^{n-1}\right) + g_{\sigma\sigma}\mathcal{A}_1^{n-1}\right)}_{P_{\sigma\sigma}^n}$$

$$+\frac{1}{6}\sigma^3 m^3 \underbrace{\left(\mathcal{M}p_{uuu}^{n-1} + 3p_{uu}^{n-1}\mathcal{M}_u + 3p_u^{n-1}\mathcal{M}_{uu} + p^{n-1}\mathcal{M}_{uuu}\right)}_{P_{\sigma\sigma\sigma}^n}$$

$$+\frac{1}{6}z_t^3 \underbrace{\left(g_{zzz}\mathcal{A}_1^{n-1} + 3g_z g_{zz}\mathcal{A}_2^{n-1} + g_z^3\mathcal{A}_3^{n-1}\right)}_{P_{zzz}^n} + \frac{1}{6}u_t^3 \underbrace{\left(g_{uuu}\mathcal{A}_1^{n-1} + 3g_u g_{uu}\mathcal{A}_2^{n-1} + g_u^3\mathcal{A}_3^{n-1}\right)}_{P_{uuu}^n}$$

$$+\frac{1}{2}z_t^2 u_t \underbrace{\left(g_{zzu}\mathcal{A}_1^{n-1} + \left(2g_z g_{zu} + g_u g_{zz}\right)\mathcal{A}_2^{n-1} + g_u g_z^2 \mathcal{A}_3^{n-1}\right)}_{P_{zzu}^n}$$

$$+\frac{1}{2}z_t u_t^2 \underbrace{\left(g_{zuu}\mathcal{A}_1^{n-1} + \left(2g_u g_{zu} + g_z g_{uu}\right)\mathcal{A}_2^{n-1} + g_z g_u^2 \mathcal{A}_3^{n-1}\right)}_{P_{zuu}^n}$$

$$+\frac{1}{2}\sigma^2 u_t \underbrace{\left(g_{u\sigma\sigma}\mathcal{A}_1^{n-1} + g_u \mathcal{A}_4^{n-1}\right)} + \frac{1}{2}\sigma^2 z_t \underbrace{\left(g_{z\sigma\sigma}\mathcal{A}_1^{n-1} + g_z \mathcal{A}_4^{n-1}\right)}$$

$P^n_{\sigma\sigma u}$ $P^n_{\sigma\sigma z}$

where

$\mathcal{A}_1^{n-1} \equiv P^{n-1}\mathcal{M}_z + \mathcal{M}P_z^{n-1}$

$\mathcal{A}_2^{n-1} \equiv P^{n-1}\mathcal{M}_{zz} + 2\mathcal{M}_z P_z^{n-1} + \mathcal{M}P_{zz}^{n-1}$

$\mathcal{A}_3^{n-1} \equiv P^{n-1}\mathcal{M}_{zzz} + 3\mathcal{M}_{zz} P_z^{n-1} + 3\mathcal{M}_z P_{zz}^{n-1} + \mathcal{M}P_{zzz}^{n-1}$

$$\begin{aligned}\mathcal{A}_4^{n-1} \equiv\ & P^{n-1}\left(\Sigma\mathcal{M}_{zuu} + \mathcal{M}_{z\sigma\sigma}\right) + \mathcal{M}\left(\Sigma P_{zuu}^{n-1} + P_{z\sigma\sigma}^{n-1}\right) \\ & +\Sigma\left(2\mathcal{M}_{zu}P_u^{n-1} + \mathcal{M}_z P_{uu}^{n-1} + \mathcal{M}_{uu}P_z^{n-1} + 2\mathcal{M}_u P_{zu}^{n-1}\right) \\ & +\mathcal{M}_{\sigma\sigma}P_z^{n-1} + P_{\sigma\sigma}^{n-1}\mathcal{M}_z + g_{\sigma\sigma}\mathcal{A}_2^{n-1}\end{aligned}$$

These formulas are derived using the *Mathematica* codes accompanying the paper.

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Table 1: Gain in computing speed from the POP method

This table compares the computing time of the one-step perturbation method to that of the POP method. The reported numbers are averages from 20 Monte Carlo replications for third order approximations. Both DSGE models are solved in Dynare++ and bond prices from the POP method are implemented in Matlab. All computations are done on an Intel Core 2 Duo P7350 PC with 3.0 GB of RAM running Windows Vista.

	5 year	10 year	15 year	20 year
Rudebusch and Swanson (2008)				
One-step perturbation method (seconds)	2.4259	10.4743	32.4616	68.8675
POP method (seconds)	0.4326	0.4536	0.4766	0.4971
Speed gain	**5.6074**	**23.0912**	**68.1106**	**138.5494**
De Paoli et al (2007)				
One-step perturbation method (seconds)	7.1842	24.5334	59.3707	111.8582
POP method (seconds)	1.7198	1.7495	1.7932	1.8311
Speed gain	**4.1775**	**14.0232**	**33.1086**	**61.0866**

Table 2: Approximation accuracy for the 10 year interest rate

The root mean sqaured errors for the approximations are computed for consumption growth at the following points: -0.1, -0.095, ..., 0.095 , 0.1. The figures in the table are multiplied by 100.

	$\gamma = 1$	$\gamma = 5$
2nd order perturbation	0.007	0.037
3rd order perturbation	0.001	0.007
2nd order consol method	0.028	0.692
3rd order consol method	0.027	0.769
1st order log-normal method	0.045	0.212
2nd order log-normal method	0.007	0.037

Table 3: Moments for the 10 year interest rate: γ=1

The 10 year interest rate is expressed in annual terms. All moments are computed based on a simulated time series of 1,000,000 observations.

	Mean	Standard deviation	Skewness	Kurtosis
2nd order perturbation	2.6724	0.1786	0.0815	3.0065
3rd order perturbation	2.6724	0.1787	0.0816	3.0113
2nd order consol method	2.8740	0.1775	0.1337	3.0214
3rd order consol method	2.8740	0.1774	0.1341	3.0332
1st order log-normal method	2.6758	0.1786	0.0000	2.9972
2nd order log-normal method	2.6724	0.1786	0.0816	3.0060
Exact solution	2.6724	0.1787	0.0818	3.0110

Table 4: Moments for the 10 year interest rate: γ=5

The 10 year interest rate is expressed in annual terms. All moments are computed based on a simulated time series of 1,000,000 observations.

	Mean	Standard deviation	Skewness	Kurtosis
2nd order perturbation	12.2038	0.8930	0.0815	3.0065
3rd order perturbation	12.2038	0.8935	0.0816	3.0113
2nd order consol method	12.4429	0.8899	0.3402	3.1520
3rd order consol method	12.4429	0.8571	0.3589	3.2342
1st order log-normal method	12.2906	0.8928	0.0000	2.9972
2nd order log-normal method	12.2039	0.8929	0.0816	3.0060
Exact solution	12.2035	0.8935	0.0818	3.0110

Figure 1: The function for the 10 year interest rate: γ=1

The x-axis reports consumption growth in deviation from the deterministic steady state. The y-axis reports the value of the 10 year interest rate in deviation from the deterministic steady state and expressed in quarterly terms.

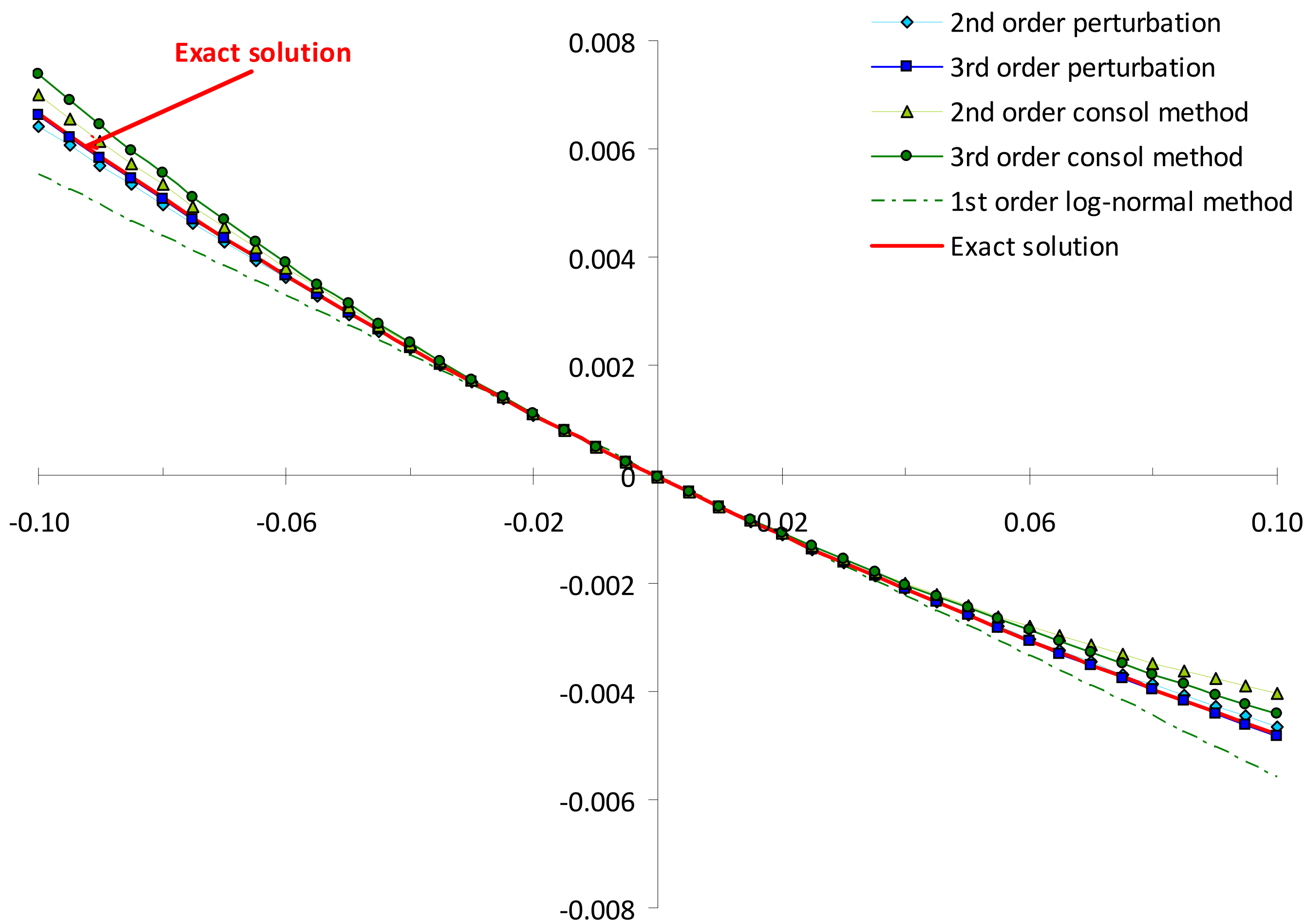

Figure 2: The function for the 10 year interest rate: γ=5

The x-axis reports consumption growth in deviation from the deterministic steady state. The y-axis reports the value of the 10 year interest rate in deviation from the deterministic steady state and expressed in quarterly terms.